COLOR


4.10 THE COLOR-MATCHING EXPERIMENT. The observer views a bipartite field and adjusts the intensities of the three primary lights to match the appearance of the test light. (A) A top view of the experimental apparatus. (B) The appearance of the stimuli to the observer. After Judd and Wyszecki, 1975.

4.11 METAMERIC LIGHTS. Two lights with these spectral power distributions appear identical to most observers and are called metamers. (A) An approximation to the spectral power distribution of a tungsten bulb. (B) The spectral power distribution of light emitted from a conventional television monitor whose three phosphor intensities were set to match the light in panel A in appearance.

FIGURE 3.2-2. Spectral energy distributions.
Algebraic formulation of color
Suppose we have some arbitrary light $C(\lambda)$. We consider that $\lambda$ is sampled e.g., 380–780 nm sampled at 2 nm intervals, so we get 201 numbers, or 400–700 nm sampled at 10 nm intervals, so 31 numbers. We can represent $C$ as

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

So we can write this as:

$$c = c_1 e_1 + c_2 e_2 + \ldots + c_n e_n$$

where the $e_i$ are the spectral (prism) colors:
\[ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \ldots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \]

Now we have some primaries. Call them \( P_1, P_2, P_3 \). Each of these is itself a column vector of length \( n \). With these primaries, we use the color matching setup to match the spectral colors.

\[ m_{1i} P_1 + m_{2i} P_2 + m_{3i} P_3 \leftrightarrow e_i \]

The symbol \( \leftrightarrow \) means "visual match." It is supposed to be used for two column vectors, and means that the two spectral distributions over wavelengths of visible light look the same to the human visual system. This equation can be written also as

\[
\begin{bmatrix}
P_1 & P_2 & P_3
\end{bmatrix}
\begin{bmatrix}
m_{1i} \\
m_{2i} \\
m_{3i}
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
0 \\
\vdots \\
0 \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

or we can write it in shorthand as \( PM_i \leftrightarrow e_i \)

where \( P \) is the \( n \times 3 \) matrix made up of the 3 column vectors of the primaries:

\[
P = \begin{bmatrix}
P_1 & P_2 & P_3
\end{bmatrix}
\]

After this matching is done for all spectral colors, you get

\[ PM \leftrightarrow I \]

This last equation is something of an abuse of the \( \leftrightarrow \) notation. When the symbol is used for matrices, we mean that the jth column on the left is a visual match for the jth column on the right. \( PM \leftrightarrow I \) is shorthand for:

\[
\begin{bmatrix}
P_1 & P_2 & P_3
\end{bmatrix}
\begin{bmatrix}
m_{11} & m_{1i} \\
m_{21} & m_{2i} \\
m_{31} & m_{3i}
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

3
The COLOR MATCHING FUNCTIONS are the rows of the color matching system matrix $M$:

![Figure 3.13 Spectral matching tristimulus curves for the CIE spectral primary system. The negative tristimulus values indicate that the colors at those wavelengths cannot be reproduced by the CIE primaries.](image)

Then for any color

$$c = c_1e_1 + c_2e_2 + \ldots + c_ne_n$$

using the axioms of "homogeneity" and additivity, we have

$$PMC \leftrightarrow c$$

If we denote $Mc = m_c$ then we have

$$Pm_c \leftrightarrow c$$

This means that with three primaries, we can match any color.

Why should 3 primaries let you match any color? Because there are 3 basic types of cones in the retina. They have different absorption characteristics as a function of wavelength:

![Relative Sensitivity vs Wavelength, nm](image)

Typical spectral absorption curves of pigments of the retina
So one can write the neural response as $r = Rc$ where this is shorthand for

$$
\begin{bmatrix}
  r_1 \\
  r_2 \\
  r_3 \\
\end{bmatrix} =
\begin{bmatrix}
  - & R_1 & - \\
  - & R_2 & - \\
  - & R_3 & - \\
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_n \\
\end{bmatrix}
$$

neural response spectral sensitivity of $i$th cone type color signal

Two colors $c_A$ and $c_B$ match if

$$
r = Rc_A = Rc_B \iff c_A \leftrightarrow c_B
$$

$$
R(c_A - c_B) = 0
$$

or $c_A - c_B$ is in the nullspace of $R$ then colors match.

$$
c \leftrightarrow \begin{bmatrix}
  P_1 & P_2 & P_3 \\
\end{bmatrix}
\begin{bmatrix}
  m_{c1} \\
  m_{c2} \\
  m_{c3} \\
\end{bmatrix} = Pm_c
$$

Since $c$ matches $Pm_c$, we have

$$
R(c - Pm_c) = 0 \quad \Rightarrow \quad Rc = RPm_c \quad \Rightarrow \quad m_c = (RP)^{-1}Rc
$$

$m_c$ is $3 \times 1$

$(RP)^{-1}$ is $3 \times 3$

$R$ is $3 \times n$

$c$ is $n \times 1$

Apply this to spectral color $e_i$

Since $PM_i \leftrightarrow e_i$ therefore $M_i = (RP)^{-1}Re_i$

Do this for all the $e_i$ and we get

$$
M = (RP)^{-1}RI = (RP)^{-1}R
$$

$(RP)^{-1}$ is a $3 \times 3$ linear transformation.
So this means that the color matching functions are within a linear transformation of our receptor responses.
Matching a color on our monitor:

We see some color $c$ and wish to reproduce this on our monitor. The neural response that we wish to replicate is $r = Rc$. Let

$$D = \begin{bmatrix} D_1 & D_2 & D_3 \end{bmatrix}$$

be the spectral power distribution of the phosphors of our display:

$$g = [g_1 g_2 g_3]^t$$

are the gun intensities. Then the neural response is

$$r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_3 & - \end{bmatrix} \begin{bmatrix} D_1 & D_2 & D_3 \end{bmatrix} g$$

We choose $g$ to match that light:

$$RDg = Rc$$

$$g = (RD)^{-1}Rc$$

$$0 \leq g \leq Max$$

constrains the colors that you can actually match.
4.14 THE XYZ STANDARD COLOR-MATCHING FUNCTIONS. In 1931 the CIE standardized a set of color-matching functions for image interchange. These color-matching functions are called $x(\lambda)$, $y(\lambda)$, and $z(\lambda)$. Industrial applications commonly describe the color properties of a light source using the three primary intensities needed to match the light source that can be computed from the XYZ color-matching functions.

Figure 2. (R,G,B)-tristimulus space (lower panel) and the (r,g,b)-chromaticity diagram (upper panel). The primary stimuli R, G, B are represented by vectors of unit length having a common origin at O. A color stimulus Q is represented by a vector whose components have lengths $R_Q$, $G_Q$, $B_Q$ along the directions defined by R, G, B. The tristimulus vector intersects the unit plane ($R + G + B = 1$) of the tristimulus space at the point Q. The unit plane contains the (r,g,b)-chromaticity diagram, in which Q represents the chromaticity of the color stimulus Q. From Wyszecki, G., & Stiles, W.S. (1982). Color science. Copyright (c) 1982 by John Wiley & Sons. Reprinted by permission of John Wiley & Sons, Inc.

Figure 3.14 Chromaticity diagram for the CIE spectral primary system. Shaded area is the color gamut of this system.

Figure 3.18 Chromaticity diagram for the NTSC receiver primary system.
Figure 10. Plot of common CRT phosphors in the CIE 1931 (x,y)-chromaticity diagram. Circles represent phosphors intended for monochrome displays whereas squares represent varieties of the P22 phosphor family, which are intended for color displays. From Laycock, J., & Viveash, J.P. (1982). Calculating the perceptibility of monochrome and color displays viewed under various illumination conditions.

(a) x-y chromaticity diagram

(b) u-v chromaticity diagram

FIGURE 3.7.2. MacAdam's ellipses of just noticeable color differences in XYZ and UVW coordinate systems (39). Axes of ellipses are 10 times actual length.
### TABLE 3.5-1. Tristimulus value conversion matrices

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<th>Output Tristimulus Value</th>
<th>Input Tristimulus Value</th>
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<th>Input Tristimulus Value</th>
</tr>
</thead>
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