Problem 6.1:

Solution: All the properties we use in this problem come from Table 9.3 in the text.

(a) By the sampling property, we have

\[ f(t) = (1 + 0^2)\delta(t) - 2(1 + 4^2)\delta(t - 4) \]
\[ = \delta(t) - 34\delta(t - 4). \]

(b) By the sampling and unit-step derivative properties, we have

\[ g(t) = \cos(2\pi t)\delta(t) + \cos(2\pi(-0.5))\delta(t + 0.5) \]
\[ = \delta(t) - \delta(t + 0.5). \]
(c) By the symmetry and scaling properties, we have

\[ h(t) = \sin(2\pi t) \times \frac{1}{2} \delta(0.25 - t) \]
\[ = \frac{1}{2} \sin(2\pi \times 0.25) \delta(t - 0.25) \]
\[ = \frac{1}{2} \delta(t - 0.25). \]

(d) By the sampling and area properties, we have

\[ y(t) = \int_{-\infty}^{\infty} (2^2 + 6) \delta(\tau - 2) \, d\tau \]
\[ = \int_{-\infty}^{\infty} 10 \delta(\tau - 2) \, d\tau \]
\[ = 10. \]

(e) Again, by the sampling and area properties, we have

\[ z(t) = \int_{6}^{\infty} (2^2 + 6) \delta(\tau - 2) \, d\tau \]
\[ = \int_{6}^{\infty} 10 \delta(\tau - 2) \, d\tau \]
\[ = 0. \]
(f) By the sampling and definite integral properties, we have

\[ a(t) = u(t + 1) + \text{rect}(1/3)\delta(t - 2) \]

\[ = u(t + 1) + \delta(t - 2). \]

(g) By the convolution property, we have

\[ b(t) = u(t - 3). \]

(h) Again, by the convolution property, we have

\[ c(t) = \triangle(t/2) - \triangle((t + 2)/2). \]
Problem 6.2:
Solution:

(a) By items 18 and 19 in Table 7.2, we have

\[ F(\omega) = 5\pi[\delta(\omega - 5) + \delta(\omega + 5)] + 3j\pi[\delta(\omega + 15) - \delta(\omega - 15)]. \]

(b) Consider

\[ x(t) = \cos^2(6t) = \frac{1 + \cos(12t)}{2}. \]

By items 14 and 18 in Table 7.1, we have

\[ X(\omega) = \pi\delta(\omega) + \pi[\delta(\omega - 12) + \delta(\omega + 12)]/2. \]

(c) By items 1 and 18 in Table 7.2, we have

\[ e^{-t}u(t) \leftrightarrow \frac{1}{1 + j\omega}, \]
\[ \cos(2t) \leftrightarrow \pi[\delta(\omega - 2) + \delta(\omega + 2)]. \]

Then, by time convolution property in Table 7.1, we have

\[ Y(\omega) = \left( \frac{1}{1 + j\omega} \right) \pi[\delta(\omega - 2) + \delta(\omega + 2)] \]
\[ = \frac{\pi}{1 + 2j}\delta(\omega - 2) + \frac{\pi}{1 - 2j}\delta(\omega + 2). \]

(d) By items 1, 15, and 18 in Table 7.2, we have

\[ 1 + \cos(3t) \leftrightarrow 2\pi\delta(\omega) + \pi[\delta(\omega + 3) + \delta(\omega - 3)], \]
\[ e^{-t}u(t) \leftrightarrow \frac{1}{1 + j\omega}. \]

Then, by the frequency convolution property in Table 7.1, we have

\[ Z(\omega) = \frac{1}{2\pi}[2\pi\delta(\omega) + \pi[\delta(\omega + 3) + \delta(\omega - 3)]] * \frac{1}{1 + j\omega} \]
\[ = \frac{1}{1 + j\omega} + \frac{1}{2(1 + j(\omega + 3))} + \frac{1}{2(1 + j(\omega - 3))}. \]
Problem 6.3:
Solution:

(a) By items 15 and 18 in Table 7.2, we have
\[ f(t) = 2 \cos(4t) + 4. \]

(b) By Euler’s formula, we have
\[ A(\omega) = 6\pi(e^{j5\omega} + e^{-j5\omega})/2. \]
Then, by item 16 in Table 7.2, we have
\[ a(t) = 3\pi[\delta(t + 5) + \delta(t - 5)]. \]

(c) By item 17 in Table 7.2, we have
\[ e^{j2nt} \leftrightarrow 2\pi\delta(\omega - 2n). \]
Thus,
\[ b(t) = \sum_{n=-\infty}^{\infty} \frac{1}{1 + n^2} e^{j2nt}. \]

Alternatively, by item 24 in Table 7.2 with \( T = \pi \), we have
\[ b(t) = \sum_{n=-\infty}^{\infty} \frac{\pi}{1 + n^2} \delta(t - n\pi). \]

You will see the equivalence of the two answers in Problem 16.

(d) Consider
\[ C(\omega) = \frac{4}{j\omega} + 4 \left( \frac{1}{j\omega} + \pi\delta(\omega) \right). \]
Then, by items 22 and 23 in Table 7.2, we have
\[ c(t) = 2\text{sgn}(t) + 4u(t) \]
\[ = 8u(t) - 2. \]

Problem 6.4:
Proof:
1. By the definition of Fourier series, we have

\[
F_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\frac{2\pi}{T} t} \, dt = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{m=-\infty}^{\infty} \delta(t - mT) e^{-jn\frac{2\pi}{T} t} \, dt = \frac{1}{T} \sum_{m=-\infty}^{\infty} \int_{-T/2}^{T/2} \delta(t - mT) e^{-jn\frac{2\pi}{T} t} \, dt = \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta(t - mT) dt = \frac{1}{T}.
\]

Thus,

\[
f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\frac{2\pi}{T} t} = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\frac{2\pi}{T} t}.
\]

2. By item 17 in Table 7.2, for each \( n \), we have

\[
\frac{1}{T} e^{jn\frac{2\pi}{T} t} \leftrightarrow \frac{2\pi}{T} \delta(\omega - n\frac{2\pi}{T}).
\]

Using the result from part (a), we have

\[
\sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\frac{2\pi}{T} t} \leftrightarrow \sum_{n=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - n\frac{2\pi}{T}).
\]

**Problem 6.5:**

**Proof:** By the result from Problem 16 part (a), we have

\[
g(t) := \sum_{n=-\infty}^{\infty} f(t) \delta(t - nT) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = f(t) \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\frac{2\pi}{T} t}.
\]

Then, using the result from Problem 16 part (b) and the frequency convolution property in Table 7.1, we have

\[
G(\omega) = \frac{1}{2\pi} F(\omega) \ast \left( \sum_{n=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - n\frac{2\pi}{T}) \right)
\]

\[
= \sum_{n=-\infty}^{\infty} F(\omega) \ast \frac{1}{T} \delta(\omega - n\frac{2\pi}{T})
\]

\[
= \sum_{n=-\infty}^{\infty} \frac{1}{T} F(\omega - n\frac{2\pi}{T}).
\]
Problem 6.6:

Proof: Let

\[ y(t) = \text{sinc} \left( \frac{\pi t}{T} \right) \ast \sum_{n=-\infty}^{\infty} f(t)\delta(t - nT). \]

We want to show \( y(t) = f(t) \). By the result of Problem 17 and the fact mentioned in the problem, we have

\[
Y(\omega) = T\text{rect} \left( \frac{\omega}{2\pi/T} \right) \cdot \left( \sum_{n=-\infty}^{\infty} \frac{1}{T} F \left( \omega - \frac{n2\pi}{T} \right) \right)
= \sum_{n=-\infty}^{\infty} \text{rect} \left( \frac{\omega}{2\pi/T} \right) F \left( \omega - \frac{n2\pi}{T} \right).
\]

Each term in the summation is a shifted version of \( F(\omega) \). If the bandwidth of \( F(\omega) \) is less than \( \pi/T \) rad/s, then

\[
\text{rect} \left( \frac{\omega}{2\pi/T} \right) F \left( \omega - \frac{n2\pi}{T} \right) = \begin{cases} F(\omega) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}
\]

Thus, \( Y(\omega) = F(\omega) \). By the inverse Fourier transform, \( y(t) = f(t) \), which recovers the sampling theorem

\[
f(t) = \text{sinc} \left( \frac{\pi t}{T} \right) \ast \sum_{n=-\infty}^{\infty} f(t)\delta(t - nT)
= \sum_{n=-\infty}^{\infty} f(nT)\text{sinc} \left( \frac{\pi}{T}(t - nT) \right).
\]