ECE 45 Discussion 1 Notes

Topics

- Steady State Analysis of RLC Circuits
- Impedance

Phasor Representation of Sinusoidal Functions:

- Phasors are used to represent sinusoidal functions and allow for easier representation of linear (resistor, capacitor, and inductor) circuits with sinusoidal voltages and currents.
- Represent \( A \cos(\omega t + \phi) \) as \( A e^{j\phi} \).
- We do not include the frequency \( \omega \) in the representation, but it is implicit.
- Differentiation and Integration:

\[
\begin{align*}
\frac{df(t)}{dt} &\quad \longleftrightarrow j\omega F \\
\int_{-\infty}^{t} f(\tau) \, d\tau &\quad \longleftrightarrow \frac{1}{j\omega} F
\end{align*}
\]

Note: We can only use phasor representation when our function (input to circuit) is sinusoidal!

Impedance:

When the inputs to our circuit are sinusoidal, we can represent resistors, capacitors, and inductors as generalized components called impedances. This allow for simple Voltage/Current relationships.

\[
\begin{align*}
v_R(t) &= i_R(t) R \quad \longleftrightarrow V_R = I_R R \\
v_C(t) &= \frac{1}{C} \int_{-\infty}^{t} i_C(\tau) \, d\tau \quad \longleftrightarrow V_C = \frac{I_C}{j\omega C} \\
v_L(t) &= L \frac{di_L(t)}{dt} \quad \longleftrightarrow V_L = j\omega L I_L
\end{align*}
\]

We can lump together the terms to end up with a general expression: \( V = I Z \) so

\[
Z_R = R, \quad Z_C = \frac{1}{j\omega C}, \quad Z_L = j\omega L.
\]

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Steady State Analysis:
Because the impedance equation \((V = I Z)\) has the same structure as Ohm’s Law \((v = i R)\), we can use circuit analysis techniques from DC circuit analysis such as:

- Parallel/Series Combinations
- KCL and KVL Analysis
- Source Transformations
- Voltage/Current Dividers
- Thevenin and Norton Equivalence

Examples:

1. **Represent the following sinusoidal function as phasors as a single complex number in rectangular form:**

   \(2 \cos(4\pi t + \pi/4) - 3 \sin(4\pi t - \pi/3)\).

   Both terms have \(\omega = 4\pi\) so we can write the function in phasor form:

   \(2 e^{j\pi/4} - 3 e^{-j5\pi/6} = 2 \left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) - 3 \left(-\frac{\sqrt{3}}{2} - \frac{j}{2}\right) = \frac{1}{2} \left(2\sqrt{2} + 3\sqrt{3} + j \left[2\sqrt{2} + 3\right]\right)\)

2. **Assume \(\omega = 2\pi\) and represent the following phasor in sinusoidal form:**

   \(\frac{1}{1 + \frac{j}{\sqrt{2}}}\).

   \(= \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}}\)

   \(e^{-j\pi/6} = \frac{1}{\sqrt{2}} e^{j\pi/4} = \frac{\sqrt{2}}{2} e^{-j5\pi/12} \leftrightarrow \frac{\sqrt{2}}{2} \cos(2\pi t - 5\pi/12)\)

3. **For what frequencies is the circuit component below purely resistive? (i.e. \(Z_{eff} = X + j0 = X\)**

   **What is the effective resistance at each frequency?**

   \[Z_{eff}(\omega) = Z_C/\left|Z_{R1} + Z_{R2} + Z_L\right| = \left(R_2 + \frac{R_1}{(R_1C\omega)^2 + 1}\right) + j \left(\omega L - \frac{R_1^2C\omega}{(R_1C\omega)^2 + 1}\right)\]
The impedance is purely resistive if the imaginary portion of $Z_{\text{eff}}(\omega)$ equals 0:

$$0 = \omega L - \frac{R_1^2 C \omega}{(R_1 C \omega)^2 + 1} = \omega^3 R_1^2 L C^2 + \omega (L - R_1^2 C) = \omega \left( \frac{1}{2} \omega^2 - 1 \right)$$

$$\therefore \omega_0 = 0, \quad \omega_1 = \sqrt{2}, \quad \omega_2 = -\sqrt{2}$$

and so the effective resistance at each frequency is:

$$Z_{\text{eff}}(0) = R_2 + R_1 = 3 \sqrt{2} \Omega$$

$$Z_{\text{eff}}(\pm \sqrt{2}) = R_2 + \frac{R_1}{2 R_1^2 C^2 + 1} = 2 \sqrt{2} \Omega$$

4. Find $i_o(t)$

Since the current and voltage sources are both of the same frequency, we can represent the voltages and currents as phasors and the resistors, capacitor, and inductor as impedances:

By Ohm’s Law, we have:

$$I_o = \frac{V_A}{Z_{R_1} + Z_L}, \quad I_2 = \frac{V - V_A}{Z_{R_2} + Z_C}$$

and by KCL we have: $I_{in} + I_2 = I_o$. By substituting in $I_o$ and $I_2$ expressions, we have

$$I_{in} + \frac{V_{in} - V_A}{Z_{R_2} + Z_C} = \frac{V_A}{Z_{R_1} + Z_L}$$

Solving for $V_A$ gives us:

$$V_A = (Z_{R_2} + Z_C) (Z_{R_1} + Z_L) \frac{I_{in} + \frac{V_{in}}{Z_{R_2} + Z_C}}{Z_{R_2} + Z_C + Z_L + Z_{R_1}}.$$
We can substitute this value of $V_A$ into our expression for $I_1$ from Ohm’s Law:

$$I_o = \frac{(Z_{R_2} + Z_C) V_{in} + V_{in}}{Z_{R_2} + Z_C + Z_L + Z_{R_1}} = \frac{-j(1-j) + 1}{1-j + j + 2} = \frac{-j}{3} = \frac{1}{3} e^{-j\pi/2}$$

Converting back to the time domain gives us:

$$i_o(t) = \frac{1}{3} \cos(2t - \pi/2) = \frac{1}{3} \sin(2t).$$

5. Let $v(t) = 2 \cos(3t + \pi/6) \text{V}, \ i(t) = \cos(3t - \pi/6) \text{A}, \ R = 2 \Omega, \ C = 1/6 \text{F}, \ and \ L = 1 \text{H}$. Determine an equivalent circuit as a voltage source in series with a resistor and an inductor.

Since the voltage and current sources are both sinusoidal of the same frequency, we can represent the circuit using phasors and impedances:

To solve for $Z_{th}$, set voltage sources to 0V (short) and current sources to 0A (open).

$$Z_{th} = Z_R / Z_C + Z_L = Z_L + \frac{Z_R Z_C}{Z_R + Z_C} = 3j + \frac{-4j}{2 - 2j} = 3j - \frac{2j}{1-j} \left(\frac{1+j}{1+j}\right) = 3j - \frac{-2 + 2j}{2} = 1 + 2j$$
To solve for $V_{th}$, leave the output open and solve for $V_{th}$:

$V = 2 e^{j\pi/6}$
$I = e^{-j\pi/6}$
$Z_R = 2$
$Z_C = -2j$
$Z_L = 3j$

Substituting the expressions for $I_R$, $I_C$, and $I_L$ into the KCL equation yields:

$$\frac{V_{th} - V}{Z_R} + \frac{V_{th}}{Z_C} + I = 0$$

Solving for $V_{th}$ gives us:

$$V_{th} = \frac{V}{Z_R} - I = \left( \frac{V}{Z_R} - I \right) (Z_C \parallel Z_R) = \frac{\left( V_{th} - Z_R \right)}{2} \frac{-4j}{2}$$

$$= \left( \frac{\sqrt{3}}{2} + \frac{j}{2} \right) \left( 1 - j \right) = j (1 - j) = 1 + j = \sqrt{2} e^{j\pi/4}$$

We could have instead solved for the short-circuit current $I_{sc}$ by connecting the terminals (shorting) at the output as follows:

$V_{th} = I_{sc} Z_{th}$. In our case:

$$I_{sc} = \frac{V_{th}}{Z_{th}} = \frac{1 + j}{1 + 2j} = \frac{(1 + j)(1 - 2j)}{5} = \frac{3 - j}{5}$$

We know $V_{th} = \sqrt{2} e^{j\pi/4}$ and $Z_{th} = 1 + 2j = R_{th} + j\omega L_{th}$, so converting to the time domain we
have

\[ R_{th} = \text{Re}\{Z_{th}\} = 1 \Omega \]
\[ L_{th} = \frac{\text{Im}\{Z_{th}\}}{\omega} = \frac{2}{3} H \]
\[ v_{th}(t) = \sqrt{2} \cos(3t + \pi/4) \]

Alternatively, we could a series of use source transformations and series and parallel combinations to solve for \(Z_{th}\) and \(V_{th}\).

\[
Z_{th} = Z_L + Z_R/Z_C = \cdots = 1 + 2j
\]
\[
V_{th} = (V/Z_R - I) (Z_R/Z_C) = \cdots = \sqrt{2}e^{j\pi/4}
\]