On the Capacity of Two-Dimensional Run-Length Constrained Channels

Akiko Kato, *Member, IEEE*, and Kenneth Zeger, *Senior Member, IEEE*

***Abstract*— Two-dimensional binary patterns that satisfy one-dimensional (d, k) run-length constraints both horizontally and vertically are considered. For a given d and k, the capacity $C_{d,k}$ is defined as $C_{d,k} = \lim_{m,n\to\infty} \log_2 N_{m,n}^{(d,k)}/mn$, where $N_{m,n}^{(d,k)}$ denotes the number of $m \times n$ rectangular patterns that satisfy the two-dimensional (d, k) run-length constraint. Bounds on $C_{d,k}$ are given and it is proven for every $d \geq 1$ and every $k > d$ that $C_{d,k} = 0$ if and only if $k = d + 1$. Encoding algorithms are also discussed.**

***Index Terms*— Channel capacity, optical storage, run-length coding, two-dimensional codes.**

I. Introduction and Main Results

A one-dimensional binary sequence is said to satisfy a (d, k)-constraint if there are at most k 0's in the row and the number of 0's between any pair of consecutive 1's is at least d. The one-dimensional *capacity* is defined as

$$E_{d,k} = \lim_{m\to\infty} \frac{\log_2 N_m^{(d,k)}}{m}$$

where $N_m^{(d,k)}$ is the number of binary patterns of length m on a line that satisfy the (d, k)-constraint. The one-dimensional capacity $E_{d,k}$ is known to be the logarithm (base 2) of the largest real root of the equation

$$X^{k+1} - X^{k-d} - X^{k-d-1} - \cdots - X - 1 = 0$$

for $0 \leq d \leq k < \infty$, and it is known that $E_{d,\infty} = E_{d-1,2d-1}$ for $d \geq 1$ (see, e.g., [1] and [10]). Therefore, for every nonnegative integer d, the one-dimensional capacity $E_{d,k}$ is positive for all $k > d$. A two-dimensional binary pattern of 0's and 1's arranged in an $m \times n$ rectangle is said to satisfy a two-dimensional (d, k)-constraint if it satisfies a one-dimensional (d, k)-constraint both horizontally and vertically. We call such patterns *valid*. The two-dimensional (d, k)-*capacity* is defined as

$$C_{d,k} = \lim_{m,n\to\infty} \frac{\log_2 N_{m,n}^{(d,k)}}{mn}$$

where $N_{m,n}^{(d,k)}$ denotes the number of valid patterns on an $m \times n$ rectangle. It is trivial to see that $C_{d,d} = 0$ for all $d \geq 0$, and hence we assume $k > d$ throughout this paper. Note that the definition of (d, k)-constraints implies monotonicity of the capacity in each variable, namely,

$$C_{d,j} \leq C_{d,k}, \qquad \text{for} \quad j \leq k \tag{1}$$

$$C_{j,k} \leq C_{d,k}, \qquad \text{for} \quad j \geq d. \tag{2}$$

Thus in particular, $\lim_{k\to\infty} C_{d,k} = C_{d,\infty}$. The two-dimensional capacity is important for certain digital recording applications, and has recently become the focus of increased study.

In this paper we derive various upper and lower bounds on $C_{d,k}$, and in particular demonstrate the curious result that for every $d \geq 1$, the two-dimensional capacity equals zero if and only if $k = d + 1$. The two-dimensional capacity has been mentioned previously in the literature, but a concise and complete proof of its existence appears to be lacking. For the sake of completeness we provide such a proof in the Appendix.

While there have been numerous studies of one-dimensional constrained codes, far fewer results have appeared concerning two-dimensional codes. Marcellin and Weber introduced *multitrack (d, k)-constrained binary codes* in [9]. In an n-track (d, k)-constrained binary code, the d-constraint is required to be satisfied one-dimensionally on each track, but the k-constraint is required to be satisfied only by the bitwise logical "or" of n consecutive tracks. Orcutt and Marcellin [15] computed capacities of *redundant* multitrack (d, k)-constrained binary codes, which allow only some fixed-size subset of the tracks (redundant tracks) to be faulty at every time instant. For the case of $d > k$, those capacity bounds were derived by Vasic [22]. Erxleben and Marcellin [4] examined error-correcting one-dimensional (d, k)-constrained binary codes for multi-track (d, k)-constrained codes; they constructed multitrack $(1, 3)$-constrained codes having better rates and better error-correcting capabilities than those previously known. Etzion [5] obtained results on mergings of two-dimensional patterns that satisfy both a (d_1, k_1)-constraint horizontally and a (d_2, k_2)-constraint vertically, and discussed the Hamming distances of such two-dimensional patterns. Weeks and Blahut [23] calculated numerical bounds on the capacity of various two-dimensional non-"run-length" constrained systems and used a Richardson extrapolation to obtain conjectures for tighter bounds.

In contrast to the one-dimensional capacity $E_{d,k}$, there is little known about the two-dimensional capacity $C_{d,k}$. It was shown by Calkin and Wilf [3] that $C_{1,\infty}$ exists and is bounded

Manuscript received March 10, 1998; revised November 11, 1998. This work was supported in part by a JSPS Fellowship for Young Scientists and by the National Science Foundation.

A. Kato is with the Department of Mathematical Engineering and Information Physics, Graduate School of Engineering, University of Tokyo, Tokyo 113-8656, Japan (e-mail: akiko@misojiro.t.u-tokyo.ac.jp).

K. Zeger is with the Department of Electrical and Computer Engineering, University of California, San Diego, CA 92103-0407 USA (e-mail: zeger@ucsd.edu).

Communicated by S. Shamai, Associate Editor for Shannon Theory.

Publisher Item Identifier S 0018-9448(99)04353-9.

as $0.587891 \leq C_{1,\infty} \leq 0.588339$. Siegel and Wolf [18] used "bit stuffing" techniques to map one-dimensional sequences onto diagonals in the plane in order to create two-dimensional (d, ∞) and $(0, k)$ constrained codes. Ashley and Marcus [2] recently discovered the surprising result that $C_{1,2} = 0$. That is, for the $(1, 2)$-constraint, effectively no positive amount of information can be stored per bit written in two dimensions. In the present paper, we generalize this result and show that $C_{d,k} = 0$ if and only if $k = d + 1$, for all $d \geq 1$. Numerous bounds are also given.

We are confident that some of the bounds in this paper can be improved upon by future researchers. Our motivations for presenting these bounds are that they are analytically aesthetic, the derivations are interesting, and in most cases no previous bounds were published or known.

The main results of this paper are Theorems 1–8 and Corollaries 1–4, which are stated below. Their proofs are given in Section II.

Theorem 1: For every positive d,

$$C_{d,d+1} = 0.$$

Theorem 2: If $k > d + 1$, then

$$C_{d,k} \geq \max_{2 \leq j \leq 1 + \frac{k-d}{2}} \left\{ \frac{\left\lfloor 1 + \frac{d}{j} \right\rfloor \log_2(j!) + \log_2(r!)}{(j+d)^2} \right\} \tag{3}$$

where $r = d \bmod j$. It can be seen from (3) (by taking $j = 2$) that $C_{d,k} > 0$ for all (d, k) such that $k \geq d + 2$, since

$$C_{d,d+2} \geq \begin{cases} \dfrac{1}{2(d+2)}, & d \text{ even} \\ \dfrac{d+1}{2(d+2)^2}, & d \text{ odd} \end{cases}$$

which is positive for all $d \geq 0$. Combining this fact with Theorem 1 gives a characterization of which (d, k)-constraints induce nonzero capacities.

Corollary 1: For every $d \geq 1$ and every $k > d$,

$$C_{d,k} = 0 \Leftrightarrow k = d + 1.$$

Fact 1. $C_{0,1} = C_{1,\infty}$.

Fact 1 holds since the two-dimensional $(1, \infty)$-constraint is equivalent to the two-dimensional $(0, 1)$-constraint, by interchanging the roles of 0 and 1. From Fact 1 and the monotonicity in (1), it immediately follows that $C_{0,k} \geq C_{1,\infty}$, for all $k \geq 2$. Theorem 3 gives a stronger lower bound on $C_{0,k}$; this lower bound approaches 1 as $k \to \infty$. The bounds in Theorems 3–5 are given in terms of the quantity $C_{1,\infty}$, whose value was determined to within ± 0.0002 by Calkin and Wilf [3].

Theorem 3: For every positive integer k,

$$C_{0,k} \geq 1 - \frac{1 - C_{1,\infty}}{\lceil k/2 \rceil}. \tag{4}$$

In [20] and [21], Talyansky, Etzion, and Roth provided an encoding algorithm for generating "conservative arrays." As a special case, their algorithm generates two-dimensional binary patterns that do not contain more than k consecutive 0's or 1's, which yields the lower bound

$$C_{0,k} \geq 1 + \frac{1}{(\lfloor k/2 \rfloor + 1)^2} \cdot \log_2\left(1 - (\lfloor k/2 \rfloor + 1) \cdot 2^{-(\lfloor k/2 \rfloor - 1)}\right). \tag{5}$$

The lower bound in (5) appeared in [19] (in Hebrew) for $k \geq 8$ and is stronger than the lower bound in (4) for all $k \geq 8$. The proof technique of Theorem 3 is, however, interesting in its own right and may lead to future ideas for improving bounds.

Theorem 4: If d and k are positive integers such that $(k + 1)/(d + 1)$ is an even integer, then

$$C_{d,k} \geq \frac{1}{d+1} - \frac{2}{k+1}(1 - C_{1,\infty}). \tag{6}$$

The inequality in (6) is valid whenever $k \equiv -1 \bmod 2(d+1)$. The right-hand side of (6) gives a lower bound on $C_{d,k'}$ for all $k' \in \{k+1, k+2, \cdots, k+2d+1\}$ by the monotonicity in (1). Thus Theorem 4 actually gives a lower bound on $C_{d,k}$ for *all* d and k.

Note that as $k \to \infty$ the lower bound in (6) approaches $C_{d,\infty} \geq 1/(d+1)$. The following theorem gives a tighter lower bound for $C_{d,\infty}$ than the limiting inequality of (6).

Theorem 5: For every $d \geq 2$,

$$C_{d,\infty} \geq \frac{C_{1,\infty}}{1 + \lfloor d/2 \rfloor}. \tag{7}$$

Theorem 6 below tightens the lower bound in Theorem 5 if and only if $d \neq 3$, but is less analytically attractive.

Theorem 6: For every $d \geq 2$, see (8) at the bottom of this page, where $r = d \bmod s$.

Corollary 2: The lower bound in Theorem 6 is stronger than the lower bound in Theorem 5 if and only if $d \neq 3$.

Theorem 7: For every positive integer k,

$$C_{0,k} \leq 1 - \left(\frac{1}{k+1}\right) \log_2\left(\frac{1}{1 - 2^{-(k+1)}}\right). \tag{9}$$

$$C_{d,\infty} \geq \max_{1 \leq s \leq d} \left\{ \frac{\left\lfloor 1 + \frac{d}{s} \right\rfloor \log_2\left(s! \sum_{i=0}^{s} \binom{s}{i} \frac{1}{i!}\right) + \log_2\left(r! \sum_{i=0}^{r} \binom{r}{i} \frac{1}{i!}\right)}{(s+d)^2} \right\} \tag{8}$$

Theorem 8: For every positive integer d,

$$C_{d,\infty} \le \frac{1}{d^2}\log_2\left(d!\sum_{i=0}^{d}\binom{d}{i}\frac{1}{i!}\right) \le \frac{1}{d}\log_2\left(\frac{2d}{e}\right)+\frac{1}{d^2}\log_2\sqrt{2\pi d}+\frac{1}{12d^3}\log_2 e.$$

The first upper bound in Theorem 8 is twice the value of the term inside the maximization in (8) when $s=d$, and becomes the trivial upper bound $C_{1,\infty}\le 1$ for $d=1$.

Note that since

$$-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n} < x+\frac{x^2}{2}+\sum_{n=3}^{\infty}x^n = x+\frac{x^2}{2}+\frac{x^3}{1-x} < x+(\ln 2)x^2$$

whenever $0<x\le(2\ln 2-1)/(2\ln 2+1)$, the lower bound in (5) implies that

$$1-C_{0,k} \le \frac{4\log_2 e}{(\lfloor k/2\rfloor+1)2^{(\lfloor k/2\rfloor+1)}}+16\cdot 2^{-2(\lfloor k/2\rfloor+1)} \le \frac{\log_2 e}{(k+1)2^{(\lfloor k/2\rfloor-2)}}+16\cdot 2^{-k-1}$$

for all sufficiently large k, which was seen in [19]. Combining this with Theorem 7, gives asymptotic bounds on how fast (as k grows) the capacity $C_{0,k}$ approaches 1 for the $(0,k)$-constraint.

Corollary 3: For sufficiently large k,

$$\frac{\frac{1}{2}\log_2 e}{(k+1)2^k} < 1-C_{0,k} \le \frac{4\sqrt{2}\cdot\log_2 e}{(k+1)2^{k/2}}+\frac{8}{2^k}. \tag{10}$$

It is interesting to note that the one-dimensional capacity $E_{0,k}$ is known to converge to one (as k grows) at the rate $(\frac{1}{4}\log_2 e)/2^k$. Corollary 4 follows from Theorems 6 and 8, and it shows that $C_{d,\infty}$ decays to zero (as d grows) exactly at the rate $(\log_2 d)/d$. The one-dimensional capacity $E_{d,\infty}$ is known to decay to zero (as d grows) exactly at the same rate $(\log_2 d)/d$.

Corollary 4:

$$\lim_{d\to\infty}\left(\frac{d}{\log_2 d}\right)\cdot C_{d,\infty}=1.$$

II. Proofs of Results

The set of integers is denoted by $\boldsymbol{Z}$, and $\boldsymbol{Z}^2$ denotes the two-dimensional integer lattice. A *two-dimensional binary code* $\mathcal{F}$ on $\boldsymbol{Z}^2$ is a set of distinct mappings f: $\boldsymbol{Z}^2\to\{0,1\}$, and each mapping is called a *codeword*. Given a codeword f, for each point $(x,y)\in\boldsymbol{Z}^2$ we call the value $f(x,y)$ the *label* of (x,y) (under f), and for any set $S\subseteq\boldsymbol{Z}^2$ the set of labels of the points of S is called the *label* of S (under f) and is denoted by $\hat{S}$. When no confusion results, the label of $\boldsymbol{Z}^2$ may be also referred to as a codeword. If all the codewords in $\mathcal{F}$ satisfy the (d,k)-constraint, we say that $\mathcal{F}$ is a (d,k)-*constrained binary code* on $\boldsymbol{Z}^2$. A subset of $\boldsymbol{Z}^2$ is called a *rectangle* if it can be written in the form

$$\{(x,y)\in\boldsymbol{Z}^2 : a\le x\le a+n-1,\ b\le y\le b+m-1\}$$

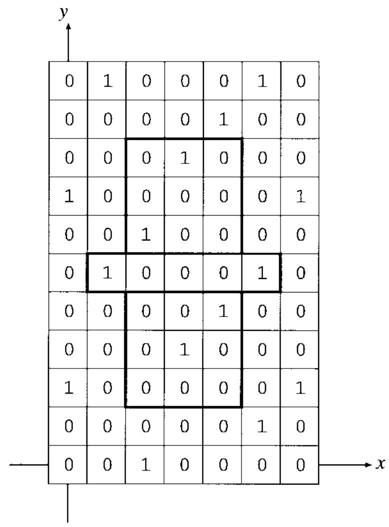

Fig. 1. Example of 3×3 adjacent matrices of $(3,5)$-constrained code on $S_{(0,0)}^{(7,11)}$.

for some integers a, b, n, m, and we denote this set by $S_{(a,b)}^{(n,m)}$. A rectangle of the form $S_{(a,b)}^{(n,n)}$ is called a *square* and is denoted by $S_{(a,b)}^{(n)}$.

Note that, given a two-dimensional binary code $\mathcal{F}$ on $\boldsymbol{Z}^2$, the label $\hat{S}$ of any square $S\subseteq\boldsymbol{Z}^2$ under $f\in\mathcal{F}$ can be viewed as a binary square matrix. Let 0^j denote j consecutive 0's. If the pattern 10^d1 occurs as a label of a horizontal line segment $\{(x,b)\in\boldsymbol{Z}^2 : a\le x\le a+d+1\}$ of length $d+2$ for some integers a and b, then we call the binary $d\times d$ square matrices $\hat{S}_{(a+1,b+1)}^{(d)}$ and $\hat{S}_{(a+1,b-d)}^{(d)}$ the *adjacent* matrices of the 10^d1 pattern. Fig. 1 shows an example of the two adjacent matrices of an occurrence of 10^d1 in a $(3,5)$-constrained binary code. Similarly, if the pattern 10^d1 occurs as a label of a vertical line segment $\{(a,y)\in\boldsymbol{Z}^2 : b\le y\le b+d+1\}$ of length $d+2$ for some integers a and b, we call the binary $d\times d$ square matrices $\hat{S}_{(a+1,b+1)}^{(d)}$ and $\hat{S}_{(a-d,b+1)}^{(d)}$ the *adjacent* matrices of the 10^d1 pattern.

A square matrix is a *permutation matrix* if there is exactly one 1 in each row and also in each column, and all other components are 0's. If all the antidiagonal components $a_{i,d-i+1}$ $(i=1,2,\cdots,d)$ are 1's in a $d\times d$ permutation matrix $A=(a_{i,j})_{i,j}$ then A is called an *anti-identity matrix*. A subset of $\boldsymbol{Z}^2$ is called a *diagonal of width* w if it can be written in the form $\{(x,y)\in\boldsymbol{Z}^2 : a\le x+y\le a+w-1\}$ for some integer a. Similarly, a subset of $\boldsymbol{Z}^2$ is called an *antidiagonal of width* w if it can be written in the form $\{(x,y)\in\boldsymbol{Z}^2 : a\le x-y\le a+w-1\}$ for some integer a.

For nonnegative integers a, b, c, we use the notation $a\equiv b \bmod c$ to indicate that $c|(a-b)$ and we use the notation $a=b \bmod c$ to mean $a\equiv b\bmod c$ and $0\le a<c$ (i.e., $a=b-\lfloor b/c\rfloor c$). Finally, for any given collection of V valid codewords on an $m\times n$ rectangle, we call the quantity $\log_2 V/(mn)$ the *coding rate* of the collection.

A. Proof of Theorem 1

It is already known that $C_{1,2}=0$ [2]. Hence we will prove Theorem 1 for $d\ge 2$ in what follows. (Our proof of Theorem 1 does not directly specialize to $d = 1$, but a slight modification does.)

Before giving the formal proof of Theorem 1 we give a brief intuitive description of the proof in order to facilitate an understanding of the rigorous details. The main idea in showing that the capacity $C_{d,k}$ is zero when $k = d + 1$ is to show that the number of valid patterns in a rectangle grows subexponentially as a function of the area of the rectangle. That is, the ratio of the growth exponent to the area of the rectangle tends to zero as the rectangle's area grows without bound. As an example, the capacity is zero if every bit of information stored in a large square requires, for example, an amount of storage space that is linear in the side length of the square, instead of constant in the side length.

Our proof of Theorem 1 first looks for any occurrence of the pattern 10^d1 in the plane and then inspects the two corresponding adjacent matrices. First it is shown that these adjacent matrices must equal each other and must be permutation matrices. Then two cases are considered: a) the adjacent matrices are neither the identity matrix nor the anti-identity matrix or b) the adjacent matrices are either the identity matrix or the anti-identity matrix. In case a) it is shown that the $(d, d + 1)$-constraint forces the label of all of $\boldsymbol{Z}^2$ to be completely determined so that there is no freedom for choosing any bits beyond the choice of the permutation matrix. In case b) it is shown that the bits that appear on any horizontal or vertical line in $\boldsymbol{Z}^2$ completely determine the rest of the choice of bits in $\boldsymbol{Z}^2$, since every occurrence of 10^d1 or $10^{d+1}1$ forces the existence of an infinite diagonal or antidiagonal of width at least d. Hence, each bit of stored information occupies an amount of area in a square that grows linearly, instead of constant, with the length of the side of the square. We conclude that the combined number of patterns that can be stored in a rectangle due to cases a) and b) is not enough to achieve positive capacity.

Conversely, to prove that the capacity is nonzero for $k \geq d+2$, we demonstrate codes that achieve nonzero coding rates.

Lemma 1. Let $d \geq 2$. For any $(d, d+1)$-constrained binary code on $\boldsymbol{Z}^2$, if the pattern 10^d1 occurs either horizontally or vertically in a codeword then its $d \times d$ adjacent matrices are permutation matrices and are equal to each other.

Proof: Without loss of generality, we can assume that the pattern 10^d1 occurs horizontally as the label of the line segment $\{(x, 0) : 0 \leq x \leq d + 1\}$. Let f: $\boldsymbol{Z}^2 \to \{0, 1\}$ be a codeword in a $(d, d + 1)$-constrained binary code on $\boldsymbol{Z}^2$ such that $f(0, 0) = f(d + 1, 0) = 1$ (and thus $f(0, y) = f(d + 1, y) = 0$ for $y = 1, 2, \cdots, d$). Therefore, for each $z \in \{1, 2, \cdots, d\}$ there must exist an $x \in \{1, 2, \cdots, d\}$ such that $f(x, z) = 1$, for otherwise 0^{d+2} would occur on the horizontal line $y = z$. Also, for each $w \in \{1, 2, \cdots, d\}$ there can be at most one $y \in \{1, 2, \cdots, d\}$ such that $f(w, y) = 1$. Hence the adjacent matrix $\hat{S}^{(d)}_{(1,1)}$ is a $d \times d$ permutation matrix.

Let σ be the unique permutation of $\{1, 2, \cdots, d\}$ such that $f(\sigma^{-1}(y), y) = 1$ for $y = 1, 2, \cdots, d$. We will show that $f(\sigma^{-1}(y), y - (d + 1)) = 1$ for all $y = 1, 2, \cdots, d$. We have $f(\sigma^{-1}(d), -1) = 1$ since $f(\sigma^{-1}(y), -1) = 0$ for $y = 1, 2, \cdots, d-1$ and $f(0, -1) = f(d+1, -1) = 0$; the former follows from $f(\sigma^{-1}(y), y) = 1$ for $y = 1, 2, \cdots, d - 1$, and the latter from $f(0, 0) = f(d+1, 0) = 1$. Thus the statement is true for $y = d$, and a straightforward induction argument shows that it is also true for $y = d - 1, d - 2, \cdots, 1$, which completes the proof. □

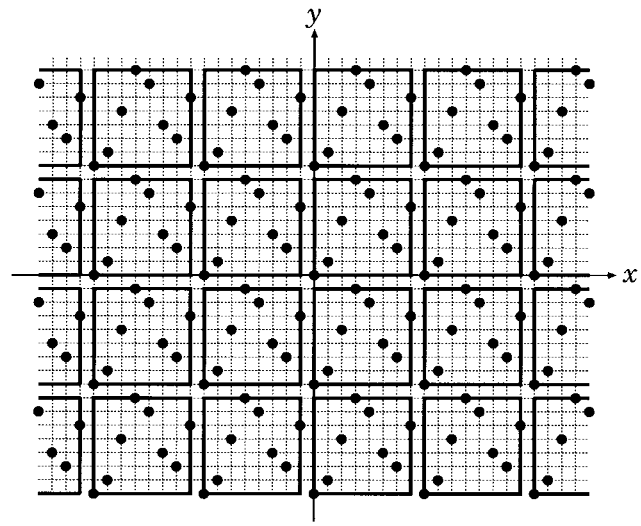

Fig. 2. Example of Lemma 2 ($d = 7$): A $(7, 8)$-constrained binary codeword whose 7×7 adjacent matrices of any 10^71 pattern are neither the identity nor the anti-identity matrix.

Lemma 2. Let $d \geq 2$. Any $(d, d + 1)$-constrained binary code on $\boldsymbol{Z}^2$ has at most $(d + 2)!$ distinct codewords that contain the pattern 10^d1, and whose $d \times d$ adjacent matrices are neither the identity matrix nor the anti-identity matrix.

Fig. 2 shows an example of the statement of the lemma, and Fig. 3 is useful for following the steps in the proof. In Fig. 2, the set $S^{(d+1)}_{(0,0)}$ and the sets whose labels are the same as $\hat{S}^{(d+1)}_{(0,0)}$ are shown as square areas surrounded by thick lines (including the boundaries), where • indicates that the label of the point is 1; otherwise the label of the point is 0. (We adopt this convention in all the figures in this paper.)

Proof: Given a $(d, d + 1)$-constrained binary code on $\boldsymbol{Z}^2$, assume that f: $\boldsymbol{Z}^2 \to \{0, 1\}$ is a codeword such that $f(0, 0) = f(d + 1, 0) = 1$ and $\hat{S}^{(d)}_{(1,1)}$ is neither the identity matrix nor the anti-identity matrix. It suffices to prove that $f(0, d+1) = f(d+1, d+1) = 1$, for then Lemma 1 forces the remainder of $\boldsymbol{Z}^2$ to be labeled in repeated patterns of adjacent matrices, i.e., the label of the whole space $\boldsymbol{Z}^2$ is uniquely determined by the label $\hat{S}^{(d+1)}_{(0,0)}$ of the square $S^{(d+1)}_{(0,0)}$.

Let σ be a permutation of $\{1, 2, \cdots, d\}$ such that $f(x, \sigma(x)) = 1$ for $x = 1, 2, \cdots, d$, (as given in the proof of Lemma 1). Either $\sigma(1) \neq d$ or $\sigma(d) \neq d$, so assume without loss of generality that $\sigma(d) \neq d$. For all $y = 1, 2, \cdots, d$ such that $y \neq \sigma(d)$, we have $f(-1, y) = 0$, since $f(\sigma^{-1}(y), y) = 1$ and $\sigma^{-1}(y) < d$ for such y's. Also, we have $f(-1, -1) = 0$, since $f(\sigma^{-1}(d), d) = f(\sigma^{-1}(d), -1) = 1$ and $\sigma^{-1}(d) < d$, and we have $f(-1, 0) = 0$, since $f(0, 0) = 1$. Therefore, $f(-1, \sigma(d)) = 1$ (for otherwise 0^{d+2} occurs vertically) and hence $f(-1, d + 1) = 0$ because $\sigma(d) \geq 1$. Together with the fact that $f(x, d+1) = 0$ for all $x \in \{1, 2, \cdots, d\}$, this implies that $f(0, d + 1) = 1$ (for otherwise 0^{d+2} occurs horizontally).

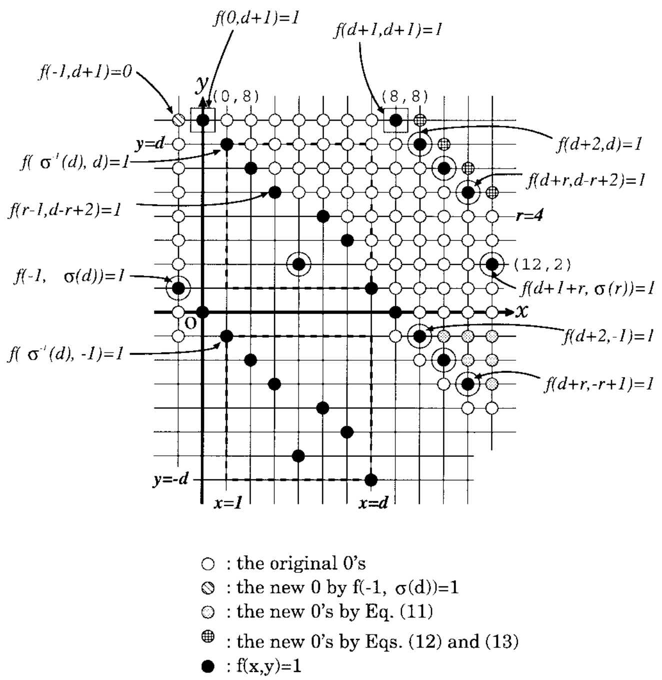

Fig. 3. Illustration of the proof of Lemma 2 for the case $d = 7,\ r = 4$.

It thus remains to be shown that $f(d+1,\ d+1) = 1$. Let $r = \min\{x \geq 1 : \sigma(x) \neq d+1-x\}$, i.e., the rth row is the first row from the top of the adjacent matrix $\hat{S}^{(d)}_{(1,1)}$ that differs from the identity matrix. If $r = 1$ then $f(d+1,\ d+1) = 1$ follows by symmetry from an analogous argument to the proof in the preceding paragraph that showed $f(0,\ d+1) = 1$. So assume $r \geq 2$.

First we show that

$$f(d+1+j,\ -j) = 1, \qquad \text{for } j = 1,\ 2,\ \cdots,\ r-1 \tag{11}$$

by induction on j. (These points are indicated in Fig. 3 as the circled black discs below the x-axis.) For $j = 1$ we have $f(d+2,\ -1) = 1$ since $f(d+2,\ y) = 0$ for $y = 0,\ 1,\ \cdots,\ d-1$ and also for $y = -2$, because $\sigma^{-1}(y) \geq 2$ for these y's, and because $f(d+1,\ 0) = 1$ and

$$f(\sigma^{-1}(d-1),\ -2) = f(\sigma^{-1}(d-1),\ d-1) = 1$$

(the last equality follows from Lemma 1 combined with the assumption $f(0,\ 0) = f(d+1,\ 0) = 1$). Now assume the induction statement is true up to and including j ($1 \leq j < r-1$). Then $f(d+1+(j+1),\ y) = 0$ for $y = d-j-1,\ d-j-2,\ \cdots,\ -j$ and for $y = -j-2$. More precisely, for $y = d-j-1,\ d-j-2,\ \cdots,\ 1$ the equality follows since $\sigma^{-1}(y) \geq j+2$ for these y's by the assumption $j < r-1$ and the definition of r; for $y = 0$ it follows since $f(d+1,\ 0) = 1$; for $y = -1,\ -2,\ \cdots,\ -j$ it follows from the induction hypothesis; for $y = -j-2$ it follows since

$$f(\sigma^{-1}(d-j-1),\ -j-2) = f(\sigma^{-1}(d-j-1),\ d-j-1) = 1$$

by Lemma 1 and $\sigma^{-1}(d-j-1) \geq j+2$. Hence we have $f(d+1+(j+1),\ -j-1) = 1$ (for otherwise 0^{d+2} appears vertically), completing the induction argument for (11).

From (11) and the definition of r we have $f(d+1+r,\ y) = 0$ for

$$y = d-(r-1),\ d-r,\ d-(r+1),\ \cdots,\ -r$$

such that $y \neq \sigma(r)$. More precisely, for

$$y = d-(r-1),\ d-r,\ \cdots,\ \sigma(r)+1,\ \sigma(r)-1,\ \cdots,\ 1$$

the equality follows since $\sigma^{-1}(y) \geq r+1$ for these y's by the definition of r; for $y = 0$ it follows since $f(d+1,\ 0) = 1$; for $y = -1,\ -2,\ \cdots,\ -(r-1)$ it follows from (11), indicated by the third type of circle in Fig. 3; for $y = -r$ it follows since $f(r+1,\ -r) = f(r+1,\ d+1-r) = 1$ by Lemma 1. Therefore, since $\sigma(r) \neq d+1-r$ by definition, we must have

$$f(d+1+r,\ \sigma(r)) = 1 \tag{12}$$

for otherwise 0^{d+2} appears vertically.

Now we will show that

$$f(d+1+r-j,\ d+1-r+j) = 1, \qquad \text{for } j = 1,\ 2,\ \cdots,\ r-1 \tag{13}$$

by induction on j. (These points are indicated in Fig. 3 as the circled black discs in the upper-right corner.) For $j = 1$ we have $f(d+r,\ d-r+2) = 1$ for otherwise 0^{d+2} would appear horizontally, since $f(x,\ d-r+2) = 0$ for $x = r,\ r+1,\ \cdots,\ d+r-1$ (because $f(r-1,\ d-r+2) = 1$) and for $x = d+r+1$ by (12). Now assume the statement is true up to and including $j-1$, for $j \geq 2$. Then $f(x,\ d-r+j+1) = 0$ for

$$x = r-j+1,\ r-j+2,\ \cdots,\ d+r-j$$

since $f(r-j,\ d-r+j+1) = 1$, and

$$f(d+r-j+2,\ d-r+j+1) = 0$$

since $f(d+r-j+2,\ d-r+j) = 1$ (by the induction hypothesis for $j-1$). Therefore, to avoid 0^{d+2} occurring horizontally, we must have

$$f(d+r-j+1,\ d-r+j+1) = 1$$

completing the induction argument.

In particular, we have $f(d+2,\ d) = 1$ so that $f(d+2,\ d+1) = 0$. Thus since $f(x,\ d+1) = 0$ for $x = 1,\ 2,\ \cdots,\ d$, we must have $f(d+1,\ d+1) = 1$ to avoid a horizontal 0^{d+2}.

It is concluded that in a $(d,\ d+1)$-constrained binary code on $\boldsymbol{Z}^2$, the number of distinct codewords f: $\boldsymbol{Z}^2 \to \{0,\ 1\}$ such that $f(0,\ 0) = f(d+1,\ 0) = 1$ and the adjacent matrix $\hat{S}^{(d)}_{(1,1)}$ is neither the identity matrix nor the anti-identity matrix is at most $d! - 2$. There are $d+1$ choices of $x \in \{0,\ 1,\ \cdots,\ d\}$ for which a codeword f: $\boldsymbol{Z}^2 \to \{0,\ 1\}$ satisfies $f(x,\ 0) = f(x+d+1,\ 0) = 1$, and there are $d+1$ choices of $y \in \{0,\ 1,\ \cdots,\ d\}$ for which $f(0,\ y) = f(0,\ y+d+1) = 1$. Therefore, a $(d,\ d+1)$-constrained binary code on $\boldsymbol{Z}^2$ can have at most $(d+1)^2(d!-2) \leq (d+2)!$ distinct codewords that contain the pattern 10^d1, and whose $d \times d$ adjacent matrices are neither the identity matrix nor the anti-identity matrix. $\square$

Note that the proof of Lemma 2 implies that if a $(d,\ d+1)$-constrained binary codeword on $\boldsymbol{Z}^2$ contains the pattern 10^d1,

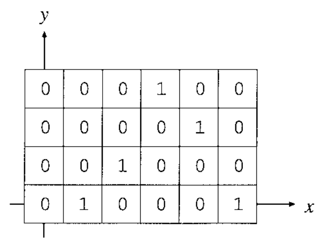

Fig. 4. Example of a $(3, 4)$-constrained binary codeword on $S_{(0,0)}^{(6,4)}$ that cannot be extended to $\boldsymbol{Z}^2$. If one more row is appended below the rectangle, it must be a binary pattern of the form $100X00$, but both $X = 0$ and $X = 1$ violate the horizontal $(3, 4)$-constraint.

and whose adjacent matrices are neither the identity matrix nor the anti-identity matrix, then it is forced to be a (d, d)-constrained codeword. In other words, such a codeword cannot contain $10^{d+1}1$ horizontally nor vertically, in spite of the $(d, d+1)$-constraint.

To determine the capacity $C_{d,k}$ we need an analog of Lemma 2 for two-dimensional $(d, d+1)$-constrained binary codes on a *bounded* rectangle rather than all of $\boldsymbol{Z}^2$. Given integers $n, m \geq 1$, a *two-dimensional* $(d, d+1)$-*constrained binary code* $\mathcal{F}$ on $S_{(0,0)}^{(n,m)}$ is a set of distinct mappings

$$f: S_{(0,0)}^{(n,m)} \to \{0, 1\}$$

that satisfy the $(d, d+1)$-constraint. That is, every codeword (i.e., the label of $S_{(0,0)}^{(n,m)}$ under every $f \in \mathcal{F}$) satisfies the $(d, d+1)$-constraint. Note that a $(d, d+1)$-constrained binary code on $S_{(0,0)}^{(n,m)}$ might not be extendible to $\boldsymbol{Z}^2$. Fig. 4 shows an example of a $(3, 4)$-constrained binary code on $S_{(0,0)}^{(6,4)}$ that cannot be extended to $\boldsymbol{Z}^2$. In [11] related nonextendable patterns are discussed.

Corollary 5. Let $d \geq 2$ *and* $n \geq m \geq 3d+3$. Any $(d, d+1)$-constrained binary code on $S_{(0,0)}^{(n,m)}$ has at most

$$(d+2)!2^{2(d+1)(n+m-2d-2)}$$

distinct codewords such that the pattern 10^d1 is contained in $S_{(d+1,d+1)}^{(n-2d-2,m-2d-2)}$ and its $d \times d$ adjacent matrices are neither the identity matrix nor the anti-identity matrix.

Proof: The number of points contained in the $m \times n$ rectangle $S_{(0,0)}^{(n,m)}$ but not contained in the

$$(m - 2(d+1)) \times (n - 2(d+1))$$

inner rectangle $S_{(d+1,d+1)}^{(n-2d-2,m-2d-2)}$ is

$$mn - (m-2(d+1))(n-2(d+1)) = 2(d+1)(n+m-2d-2).$$

There are at most $2^{2(d+1)(n+m-2d-2)}$ labels of these points. From Lemma 2 (whose proof does not use points farther than $d+1$ from the inner rectangle) there are at most $(d+2)!$ valid labels of the inner rectangle, completing the proof of the corollary. $\square$

Fig. 5 shows an example of Corollary 5 for $n = 23$, $m = 19$, $d = 3$.

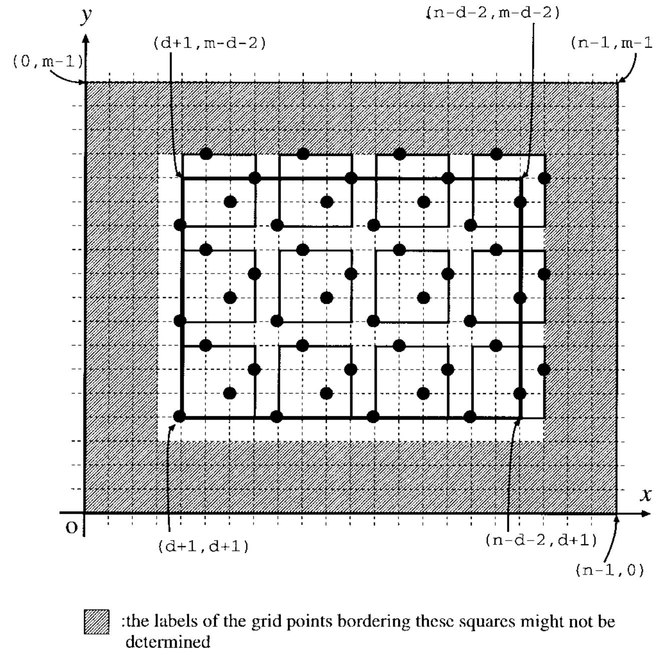

Fig. 5. Example of Corollary 5 ($n = 23$, $m = 19$, $d = 3$).

Lemma 3. Let $d \geq 2$. For any $(d, d+1)$-constrained binary code on $\boldsymbol{Z}^2$, if the pattern 10^d1 occurs in a codeword and its $d \times d$ adjacent matrices are identity (respectively, anti-identity) matrices then the 10^d1 pattern is contained in an infinite diagonal (respectively, antidiagonal) of width $d+2$.

Proof: Let $f: \boldsymbol{Z}^2 \to \{0, 1\}$ be a codeword in a $(d, d+1)$-constrained binary code on $\boldsymbol{Z}^2$ such that $f(0, 0) = f(d+1, 0) = 1$ and $\hat{S}_{(1,1)}^{(d)}$ is the identity matrix. It suffices to show that $f(-1, 1) = f(0, d+1) = 1$. Since $f(-1, y) = 0$ for $y = 2, 3, \cdots, d$ and also for $y = 0$ and $y = -1$, we have $f(-1, 1) = 1$. Therefore, $f(0, d+1) = 1$ because $f(-1, d+1) = 0$ and $f(x, d+1) = 0$ for $x = 1, 2, \cdots, d$. $\square$

Fig. 6 shows an example of Lemma 3 for $d = 7$, and the proof is also illustrated in Fig. 7.

Corollary 6. Let $d \geq 2$. For any $(d, d+1)$-constrained binary code on $\boldsymbol{Z}^2$, a codeword cannot have occurrences of both the identity and the anti-identity matrices as adjacent matrices of different 10^d1 patterns.

Proof: In light of Lemma 3, if the pattern 10^d1 occurs on $\boldsymbol{Z}^2$ with its adjacent matrices being the identity matrix, and the pattern 10^d1 also occurs somewhere else on $\boldsymbol{Z}^2$ with its adjacent matrices being the anti-identity matrix, then the diagonal strip of width $d+2$ that contains the former 10^d1 pattern and the antidiagonal strip of width $d+2$ that contains the latter 10^d1 pattern intersect somewhere on $\boldsymbol{Z}^2$. This is a contradiction. $\square$

Remark 1. Let $d \geq 2$. In view of the proof of Lemma 2, for any $(d, d+1)$-constrained binary code on $\boldsymbol{Z}^2$, a codeword cannot have occurrences of both the identity matrix and any other matrix as adjacent matrices of different 10^d1 patterns. Similarly, a codeword cannot have occurrences of both the anti-identity matrix and any other matrix as adjacent matrices of different 10^d1 patterns.

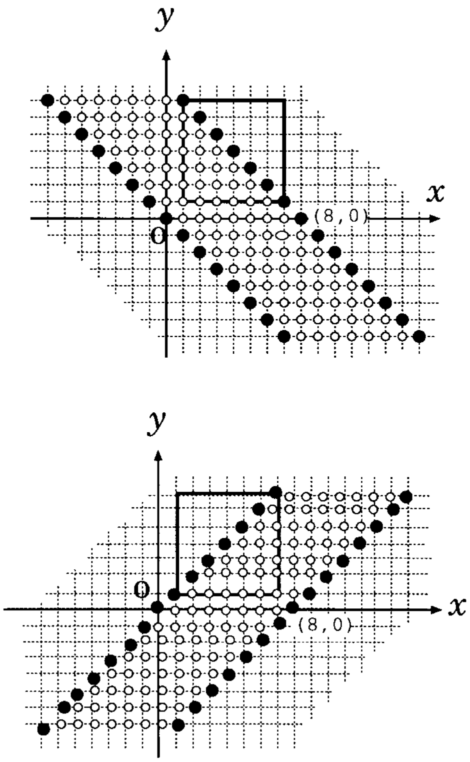

Fig. 6. Example of Lemma 3 ($d = 7$).

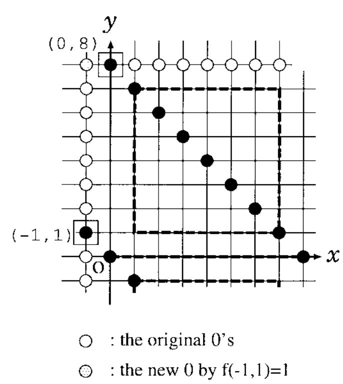

Fig. 7. Illustration of the proof of Lemma 3 for the case $d = 7$.

Lemma 4. Let $d \geq 2$. Suppose that a codeword in a $(d, d{+}1)$-constrained binary code on $\boldsymbol{Z}^2$ has the property that every 10^d1 pattern has identity (anti-identity) matrices as its adjacent matrices. Then, given the label of any infinite horizontal strip of width 1, the codeword is uniquely determined.

Proof: Without loss of generality assume we are given the label of the integers on the x-axis and that every occurrence of 10^d1 induces (via Lemma 3) a diagonal of width $d+2$. Further assume without loss of generality that the sequence $10^d10^{d+1}1$ occurs on the horizontal line segment from $(0, 0)$ to $(2d{+}3, 0)$. The sequence 10^d1 induces a diagonal of width $d+2$, and therefore the sequence $10^{d+1}1$ induces a diagonal of width $d+3$. This follows because each 1 on the diagonal $\{(y, -y) : \quad y \in \boldsymbol{Z}\}$ forces $d{+}1$ 0's to the right (if only d 0's followed then another diagonal of width $d+2$ would result, forcing a 1 in position $(2d+2, 0)$). This argument can easily be extended using induction to show that all occurrences of $10^{d+1}1$ on the x-axis induce diagonals of width $d+3$ (see Fig. 8). □

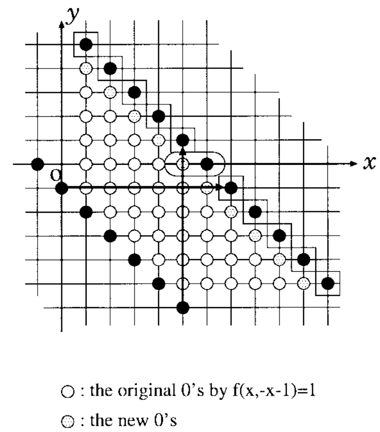

Fig. 8. Illustration of the proof of Lemma 4 for the case $d = 5$.

The following corollary is an analog of Lemma 4 for $(d, d+1)$-constrained binary codes on $S_{(0,0)}^{(n,m)}$.

Corollary 7. Let $d \geq 2$ and $n \geq m \geq 3d+3$. Given a $(d, d+1)$-constrained binary code on $S_{(0,0)}^{(n,m)}$, suppose that a codeword has the property that every 10^d1 pattern in $S_{(d+1,d+1)}^{(n-2d-2,\, m-2d-2)}$ has identity (anti-identity) matrices as its adjacent matrices. Then, given the labels of the horizontal line segments

$$L_1 = \{(x, d+1) \in S_{(0,0)}^{(n,m)} : 0 \leq x \leq n-1\}$$

and

$$L_2 = \{(x, m-d-2) \in S_{(0,0)}^{(n,m)} : 0 \leq x \leq n-1\}$$

the label of $S_{(d+1,d+1)}^{(n-2d-2,\, m-2d-2)}$ in the codeword is uniquely determined.

Proof: In view of Lemma 4, if the label of L_1 is given then the label of the set

$$S = \{(x, y) \in S_{(d+1,d+1)}^{(n-2d-2,\, m-2d-2)} : d+1 \leq x+y \leq n+d\}$$

is uniquely determined. Furthermore, if the label of the set $L_2 \setminus S$ is given and is consistent with the label of S under the $(d, d+1)$-constraint, then it additionally determines the label of the set

$$S = \{(x, y) \in S_{(d+1,d+1)}^{(n-2d-2,\, m-2d-2)} : n+d \leq x+y \leq m+n-d-3\}.$$

This completes the proof. □

Fig. 9 shows an example of Corollary 7 for $n = 23$, $m = 19$, $d = 3$.

Corollary 8. Let $d \geq 2$ and $n \geq m \geq 3d+3$. Any $(d, d+1)$-constrained binary code on $S_{(0,0)}^{(n,m)}$ has at most $2^{\frac{n+m}{d+1}+2(d+1)(n+m-2d-2)}$ distinct codewords in which every 10^d1 pattern in $S_{(d+1,d+1)}^{(n-2d-2,\, m-2d-2)}$ has the identity (anti-identity) matrix as its adjacent matrices.

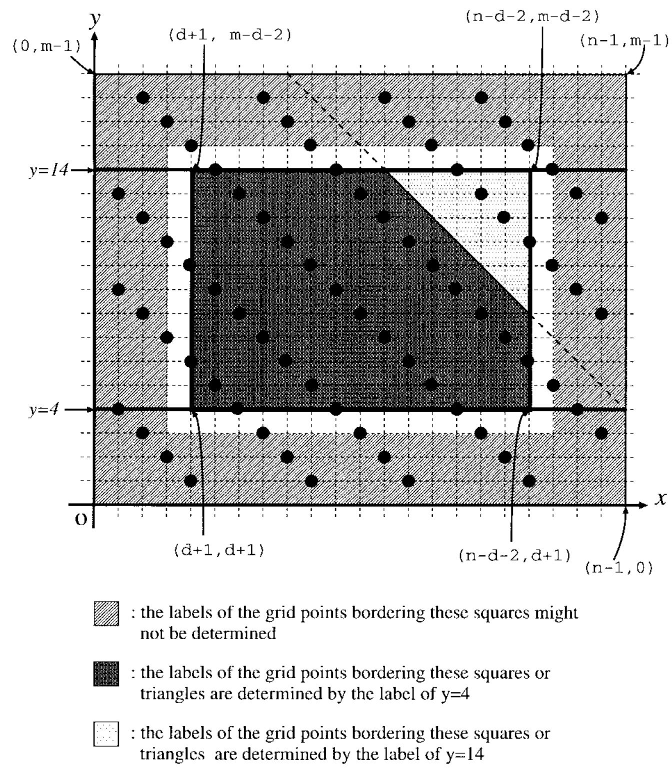

Fig. 9. Example of Corollary 7 ($n = 23$, $m = 19$, $d = 3$).

Proof: Immediately follows from Corollary 7, since we have $(n+m)/(d+1)$ choices for the labels of the horizontal line segments L_1 and L_2, and also in $S_{(0,0)}^{(n,m)}$, there are at most $2(d+1)(n+m-2d-2)$ grid points in the rectangular annulus of width d, as in the proof of Corollary 5. $\square$

Lemma 5. Let $d \geq 2$. Any $(d, d+1)$-constrained binary code on $\boldsymbol{Z}^2$ has at most $(d+2)!$ codewords that do not contain the pattern 10^d1.

Proof: Any codeword in a $(d, d+1)$-constrained binary code that does not contain the pattern 10^d1 on $\boldsymbol{Z}^2$ is uniquely determined by the label $\hat{S}_{(0,0)}^{(d+2)}$ of a $(d+2)\times(d+2)$ square, which must be a permutation matrix. There are $(d+2)!$ such matrices. $\square$

The following corollary immediately follows from the proof of Lemma 5 for $(d, d+1)$-constrained binary codes on $S_{(0,0)}^{(n,m)}$, since 0 or 1 can be chosen arbitrarily for at most $2(d+1)(n+m-2d-2)$ locations in $S_{(0,0)}^{(n,m)}$.

Corollary 9. Let $d \geq 2$ and $n \geq m \geq 3d+3$. Any $(d, d+1)$-constrained binary code on $S_{(0,0)}^{(n,m)}$ has at most $(d+2)!2^{2(d+1)(n+m-2d-2)}$ distinct codewords that do not contain the pattern 10^d1 in $S_{(d+1,d+1)}^{(n-2d-2,\, m-2d-2)}$.

Proof of Theorem 1: For sufficiently large n and m, Remark 1 holds for any $(d, d+1)$-constrained binary code on the bounded rectangle $S_{(0,0)}^{(n,m)}$. Therefore, combining Corollaries 5, 8, and 9, any $(d, d+1)$-constrained binary code on $S_{(0,0)}^{(n,m)}$ has at most

$$((d+2)! + (d+2)! + 2^{\frac{n+m}{d+1}+1})2^{2(d+1)(n+m-2d-2)}$$

distinct codewords; this number is smaller than $2^{3(d+1)(n+m)}$ for sufficiently large n, m. Hence we have

$$\begin{aligned} C_{d,\,d+1} &= \lim_{m,\,n\to\infty} \frac{\log_2 N_{m,\,n}^{(d,\,d+1)}}{mn} \\ &\leq \lim_{m,n\to\infty} \frac{3(d+1)(n+m)}{mn} = 0. \qquad \square \end{aligned}$$

B. Proof of Theorem 2

Lemma 6: Let d and k' be nonnegative integers such that $k'-d$ is positive and even. Let $a = (k'+d+2)/2$, $b = (k'-d+2)/2$, $r = a \bmod b$, and let $\mathcal{V}_{d,\,k'}$ be the set of all block-diagonal $a\times a$ matrices of the form

$$\begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \ddots & & \\ & & & A_{\lfloor a/b \rfloor} & \\ & & & & A' \end{bmatrix}$$

where each A_i is a $b\times b$ permutation matrix, A' is an $r\times r$ permutation matrix, and all the elements not specified are 0's. If a mapping f: $\boldsymbol{Z}^2 \to \{0, 1\}$ satisfies $\hat{S}_{(x,\,y)}^{(a)} \in \mathcal{V}_{d,\,k'}$ whenever $x \equiv y \equiv 0 \bmod a$, then f is a (d, k')-constrained codeword.

Proof: By symmetry, it suffices to show that if $\hat{S}_{(0,0)}^{(a)} \in \mathcal{V}_{d,\,k'}$ then $\hat{S}_{(a,0)}^{(a)}$ can be any element of $\mathcal{V}_{d,\,k'}$ without violating the (d, k')-constraint. If

$$\hat{S}_{(0,0)}^{(a)},\ \hat{S}_{(a,0)}^{(a)} \in \mathcal{V}_{d,\,k'}$$

then for $1 \leq i \leq l$ the label of the rectangle $S_{(0,\,a-bi)}^{(2a,\,b)}$ (i.e., rows $b(i-1)+1$ through bi of $\hat{S}_{(0,0)}^{(2a,\,a)}$), is of the form

$$\hat{S}_{(0,\,a-bi)}^{(2a,\,b)} = \left[\boxed{O_{b(i-1)}} \ \boxed{P} \ \boxed{O_d} \ \boxed{P'} \ \boxed{O_{d-b(i-1)}} \right]$$

where O_j is the $b\times j$ rectangular matrix, all of whose elements are 0's, and P and P' are $b\times b$ permutation matrices. Therefore, the number of consecutive 0's between any pair of two horizontally consecutive 1's in $S_{(0,\,a-ib)}^{(2a,\,b)}$ is at least d and at most $(b-1)+d+(b-1)=k'$. Similarly, the label of the rectangle $S_{(0,0)}^{(2a,\,r)}$ is of the form

$$\hat{S}_{(0,0)}^{(2a,\,r)} = \left[\boxed{O_{b\lfloor a/b \rfloor}} \ \boxed{P} \ \boxed{O_{a-r}} \ \boxed{P'} \right]$$

where P and P' are $r\times r$ permutation matrices. Therefore, the number of consecutive 0's between any pair of two horizontally consecutive 1's in $S_{(0,0)}^{(2a,\,r)}$ is at least

$$a - r \geq a - (b-1) = d+1$$

and at most

$$(r-1)+(a-r)+(r-1) = a+r-2 \leq a+b-3 = k'-1.$$

This completes the proof. $\square$

Proof of Theorem 2: Let $\mathcal{F}$ be a (d, k')-constrained binary code on $\boldsymbol{Z}^2$ consisting of all mappings f given in the statement of Lemma 6. Given integers n and m, we define a (d, k')-constrained binary code $\mathcal{F}_{(0,0)}^{(n,m)}$ on $S_{(0,0)}^{(n,m)}$ as the restriction of $\mathcal{F}$ to $S_{(0,0)}^{(n,m)}$; i.e., each codeword in $\mathcal{F}_{(0,0)}^{(n,m)}$ is the restriction of a codeword in $\mathcal{F}$ to $S_{(0,0)}^{(n,m)}$. Then, since $a - b = d$, we have

$$C_{d,k'} \geq \lim_{m,n\to\infty} \frac{\log_2 |\mathcal{F}_{(0,0)}^{(n,m)}|}{mn} = \frac{\log_2 |\mathcal{V}_{d,k'}|}{a^2} = \frac{\log_2 \left((b!)^{\lfloor a/b \rfloor}(r!)\right)}{a^2} = \frac{\left\lfloor \frac{b+d}{b} \right\rfloor \log_2(b!) + \log_2(r!)}{(b+d)^2}.$$

Since $C_{d,k} \geq C_{d,k'}$ for all $k' \leq k$, we have

$$C_{d,k} \geq \max_{\substack{d+2 \leq k' \leq k,\\ k'-d \text{ even}}} \frac{\left\lfloor \frac{b+d}{b} \right\rfloor \log_2(b!) + \log_2(r!)}{(b+d)^2} = \max_{2 \leq b \leq 1+\frac{k-d}{2}} \frac{\left\lfloor \frac{b+d}{b} \right\rfloor \log_2(b!) + \log_2(r!)}{(b+d)^2}. \tag{14}$$

The range of k' in (14) is restricted to $k' \geq d+2$ instead of $k' \geq d$ because $C_{d,d} = 0$ and $k' - d$ must be even. $\square$

Recall that the coding rate of any (d, k)-constrained binary code $\mathcal{F}_{(0,0)}^{(n,m)}$ on $S_{(0,0)}^{(n,m)}$ is defined as

$$\frac{\log_2 |\mathcal{F}_{(0,0)}^{(n,m)}|}{mn}$$

where $|\mathcal{F}_{(0,0)}^{(n,m)}|$ is the number of codewords in $\mathcal{F}_{(0,0)}^{(n,m)}$. We will see in the following sections that the limits of the coding rates of certain sequences of binary codes will also establish the lower bounds in Theorems 3–6, as was done in the proof of Theorem 2.

C. Proof of Theorem 3

We construct $(0, k)$-constrained binary codes on $m \times n$ rectangles, whose coding rates approach the lower bound in (4) as $n, m \to \infty$. Fig. 10 illustrates this construction for a $(0, 5)$-constrained binary codeword on $S_{(0,0)}^{(12,9)}$, i.e., $k = 5$, $m = 9$, and $n = 12$.

Intuitively, if k is odd then such a sequence of $(0, k)$-constrained binary codes is constructed as follows. Let n and m be positive integers that are divisible by $(k+1)/2$ and assume $n \geq m$. Let $n' = \frac{n}{(k+1)/2}$ and $m' = \frac{m}{(k+1)/2}$. First, we construct $(0, 1)$-constrained binary codewords $f_0, f_1, \cdots, f_{(k-1)/2}$ on $S_{(0,0)}^{(n',m')}$. Then, for each codeword f_t, we replace its bits by $1 \times (k+1)/2$ rectangles containing the bit patterns

$$\underbrace{X \cdots X}_{t} 0 \underbrace{X \cdots X}_{(k-1)/2-t} \quad \text{and} \quad \underbrace{X \cdots X}_{t} 1 \underbrace{X \cdots X}_{(k-1)/2-t}$$

where each X is arbitrarily chosen from $\{0, 1\}$. That is, each such rectangle has $(k-1)/2$ "free bits" and one fixed 0 or one fixed 1. Finally, we merge these $(k+1)/2$ codewords on $S_{(0,0)}^{(n',m)}$ into a single codeword on the original $m \times n$ rectangle $S_{(0,0)}^{(n,m)}$, by interlacing; i.e., regarding the ith row of f_t as the $\left(\frac{(k+1)(i-1)}{2} + t + 1\right)$th row of the resulting codeword on $S_{(0,0)}^{(n,m)}$, for $i \in \{1, 2, \cdots, m'\}$. (The rows are ordered from bottom to top.)

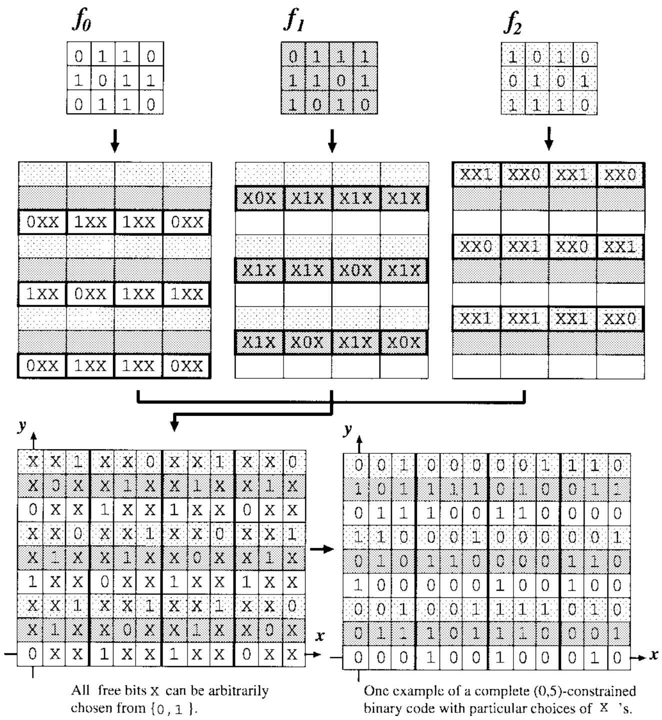

Fig. 10. A $(0, 5)$-constrained binary codeword $g_H^{(12,9)}$ on $S_{(0,0)}^{(12,9)}$ constructed from three $(0, 1)$-constrained binary codewords f_0, f_1, and f_2 on $S_{(0,0)}^{(4,3)}$. See Theorem 3.

Formally, let $\mathcal{F}^{(n',m')}$ be a $(0, 1)$-constrained binary code on $S_{(0,0)}^{(n',m')}$, let

$$H = \{f_0, f_1, \cdots, f_{(k-1)/2}\}$$

be a set of $(k+1)/2$ codewords in $\mathcal{F}^{(n',m')}$, and let

$$g_H^{(n,m)}\colon \quad S_{(0,0)}^{(n,m)} \to \{0, 1\}$$

be any mapping such that the label of the rectangle $S_{(x,y)}^{((k+1)/2,1)}$ under $g_H^{(n,m)}$ satisfies

$$\hat{S}_{(x,y)}^{((k+1)/2,1)} = \begin{cases} X^t 0 X^{\frac{k-1}{2}-t} & \text{if } f_t\left(\frac{x}{(k+1)/2}, \frac{y-t}{(k+1)/2}\right) = 0 \\ X^t 1 X^{\frac{k-1}{2}-t} & \text{if } f_t\left(\frac{x}{(k+1)/2}, \frac{y-t}{(k+1)/2}\right) = 1 \end{cases} \tag{15}$$

for

$$(x, y) \equiv (0, t) \bmod \frac{(k+1)}{2}$$

for each $t = 0, 1, \cdots, (k-1)/2$, and where each X is an arbitrary choice of either 0 or 1.

The resulting codeword $g_H^{(n,m)}$ is shown to satisfy the $(0, k)$-constraint on $S_{(0,0)}^{(n,m)}$ as follows. It is straightforward from the definition that $g_H^{(n,m)}$ satisfies the $(0, k)$-constraint horizontally; the number of consecutive 0's between any pair of two horizontally consecutive 1's is at most $((k-1)/2) - t + ((k+1)/2) + t = k$, since each f_t satisfies the $(0, 1)$-constraint horizontally (this is achieved if 101 maps to $0^t 10^{((k-1)/2)-t} 0^{((k+1)/2)} 0^t 10^{((k-1)/2)-t}$). A similar argument shows that $g_H^{(n,m)}$ satisfies the $(0, k)$-constraint vertically.

Proof of Theorem 3: First, suppose that k is odd. Let n, m, n', m' be positive integers such that $\frac{k+1}{2}|n$, $\frac{k+1}{2}|m$, and $n \geq m$, and let $n' = \frac{n}{(k+1)/2}$ and $m' = \frac{m}{(k+1)/2}$. Let $\mathcal{G}_H^{(n,m)}$ be the set of all mappings $g_H^{(n,m)}$ defined above, and let

$$\mathcal{G}^{(n,m)} = \bigcup_{H \subseteq \mathcal{F}^{(n',m')}:\, |H| = \frac{k+1}{2}} \mathcal{G}_H^{(n,m)},$$

a $(0, k)$-constrained binary code on $S_{(0,0)}^{(n,m)}$. We will show that the coding rate of $\mathcal{G}^{(n,m)}$ approaches the lower bound in (4) as $n, m \to \infty$, for a particular choice of a $(0, 1)$-constrained code $\mathcal{F}^{(n',m')}$ on $S_{(0,0)}^{(n',m')}$. Let

$$\{\mathcal{F}^{(n',m')}\}_{n',m'=1}^{\infty}$$

be a sequence of $(0, 1)$-constrained binary codes on $S_{(0,0)}^{(n',m')}$ such that the coding rate $\log_2 |\mathcal{F}^{(n',m')}|/(n'm')$ of $\mathcal{F}^{(n',m')}$ approaches $C_{0,1}$ as $n', m' \to \infty$ (such codes were shown to exist in [3]). $\frac{k+1}{2}$ patterns are interlaced, each chosen from among $|\mathcal{F}^{(n',m')}|$ possible patterns. Each resulting interlaced $m \times n$ rectangular pattern has $\frac{k-1}{k+1}mn$ free bits. Thus the total number of valid $m \times n$ patterns created with this construction is

$$|\mathcal{G}^{(n,m)}| = |\mathcal{F}^{(n',m')}|^{((k+1)/2)} \cdot 2^{((k-1)/(k+1))mn}$$

and hence

$$\begin{aligned}\lim_{m,n\to\infty} & \frac{\log_2 |\mathcal{G}^{(n,m)}|}{mn} \\ &= \frac{k+1}{2} \lim_{m,n\to\infty} \left(\frac{\log_2 |\mathcal{F}^{(n',m')}|}{m'n'} \cdot \frac{m'n'}{mn}\right) + \frac{k-1}{k+1} \\ &= \frac{2}{k+1} C_{0,1} + \frac{k-1}{k+1}.\end{aligned}$$

Together with Fact 1, this completes the proof of Theorem 3 for odd k.

For even k, Theorem 3 immediately follows from the monotonicity in (1), and Theorem 3 for $k-1$ odd. $\square$

D. Proof of Theorem 4

Theorem 4 follows directly from Theorem 3 and (16) in Lemma 7 below (using k' odd).

Lemma 7: Let d, k, d', k' be nonnegative integers such that

$$\frac{d+1}{d'+1} = \frac{k+1}{k'+1}$$

is an integer. Then

$$C_{d,k} \geq \frac{d'+1}{d+1} C_{d',k'}.$$

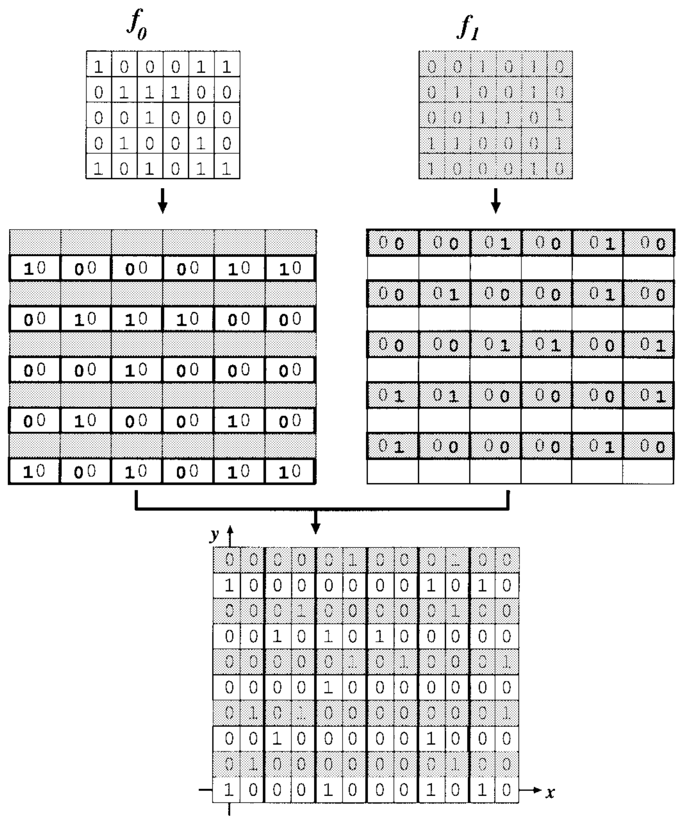

Fig. 11. A $(1, 7)$-constrained binary codeword $g_H^{(12,10)}$ on $S_{(0,0)}^{(12,10)}$ constructed from two $(0, 3)$-constrained binary codewords on $S_{(0,0)}^{(6,5)}$. See Theorem 4 and Lemma 7.

In particular, when $d' = 0$ this implies

$$C_{d,k} \geq \frac{1}{d+1} C_{0,k'} \tag{16}$$

where

$$k' = \frac{k+1}{d+1} - 1.$$

Proof of Lemma 7: Let

$$s = \frac{d+1}{d'+1} = \frac{k+1}{k'+1}.$$

To establish Lemma 7, we construct (d, k)-constrained binary codes on $m \times n$ rectangles, whose coding rates approach $(1/s)C_{d',k'}$ as $n, m \to \infty$. Fig. 11 illustrates this construction for a $(1, 7)$-constrained binary codeword on $S_{(0,0)}^{(12,10)}$ from two $(0, 3)$-constrained binary codeword on $S_{(0,0)}^{(6,5)}$, i.e., $d = 1$, $k = 1$, $d' = 0$, $k' = 3$, $s = 2$, $m = 10$, and $n = 12$.

Let n and m be positive integers that are divisible by s, assume $n \geq m$, and let $n' = n/s$ and $m' = m/s$. A (d, k)-constrained binary codeword on $S_{(0,0)}^{(n,m)}$ is constructed from a set of s (d', k')-constrained binary codewords on $S_{(0,0)}^{(n',m')}$ as follows. Let $\mathcal{F}^{(n',m')}$ be a (d', k')-constrained binary code on $S_{(0,0)}^{(n',m')}$, and let $H = \{f_0, f_1, \cdots, f_{s-1}\}$ be a set of s codewords in $\mathcal{F}^{(n',m')}$. For each codeword f_t, we replace every occurrence of 0 and 1 by a $1 \times s$ rectangle containing the bit patterns 0^s and 0^t10^{s-1-t}, respectively. Then, we merge these s codewords into a new large codeword, on the original $m \times n$ rectangle $S_{(0,0)}^{(n,m)}$, by regarding the ith row of f_t as the $(s(i-1)+t+1)$th row of the resulting codeword for $i \in \{1, 2, \cdots, m'\}$ (the rows are ordered from bottom to top). Formally, let $g_H^{(n,m)}\colon S_{(0,0)}^{(n,m)} \to \{0, 1\}$ be any mapping such that the label of $S_{(x,y)}^{(s,1)}$ under $g_H^{(n,m)}$ satisfies

$$\hat{S}_{(x,y)}^{(s,1)} = \begin{cases} 0^s, & \text{if } f_t\left(\dfrac{x}{s}, \dfrac{y-t}{s}\right) = 0 \\ 0^t10^{s-1-t}, & \text{if } f_t\left(\dfrac{x}{s}, \dfrac{y-t}{s}\right) = 1 \end{cases}$$

for $(x, y) \equiv (0, t) \bmod s$, for each $t = 0, 1, \cdots, s-1$.

The resulting codeword $g_H^{(n,m)}$ is shown to satisfy the (d, k)-constraint on $S_{(0,0)}^{(n,m)}$ as follows. Since each f_t satisfies the (d', k')-constraint horizontally by definition, the number of consecutive 0's between any pair of two horizontally consecutive 1's in $g_H^{(n,m)}$ is at least $s-1-t+d's+t=d$ and at most $s-1-t+k's+t=k$; the former is achieved since the pattern $10^{d'}1$ maps to

$$0^t10^{s-1-t}\underbrace{0^s\cdots0^s}_{d'}0^t10^{s-1-t}$$

and the latter is achieved since the pattern $10^{k'}1$ maps to

$$0^t10^{s-1-t}\underbrace{0^s\cdots0^s}_{k'}0^t10^{s-1-t}.$$

A similar argument shows that $g_H^{(n,m)}$ satisfies the (d, k)-constraint vertically.

Let $\mathcal{G}_H^{(n,m)}$ be the set of all mappings $g_H^{(n,m)}$, and let

$$\mathcal{G}^{(n,m)} = \bigcup_{H \subseteq \mathcal{F}^{(n',m')}:\, |H|=s} \mathcal{G}_H^{(n,m)},$$

a (d, k)-constrained binary code on $S_{(0,0)}^{(n,m)}$. We show that the coding rate of $\mathcal{G}^{(n,m)}$ approaches the lower bound in Lemma 7 as $n, m \to \infty$, for a particular choice of a (d', k')-constrained binary code $\mathcal{F}^{(n',m')}$ on $S_{(0,0)}^{(n',m')}$. Let

$$\{\mathcal{F}^{(n',m')}\}_{n',m'=1}^{\infty}$$

be a sequence of (d', k')-constrained binary codes on $S_{(0,0)}^{(n',m')}$ such that the coding rate $\log_2|\mathcal{F}^{(n',m')}|/(n'm')$ of $\mathcal{F}^{(n',m')}$ approaches $C_{d',k'}$ as $n', m' \to \infty$. Note that such a sequence exists because of the definition of the capacity $C_{d',k'}$. Then, we have

$$\begin{aligned} \lim_{m,n\to\infty} \frac{\log_2|\mathcal{G}^{(n,m)}|}{mn} &= \lim_{m,n\to\infty} \frac{\log_2|\mathcal{F}^{(n',m')}|^s}{mn} \\ &= s \lim_{m,n\to\infty} \frac{\log_2|\mathcal{F}^{(n',m')}|}{m'n'} \cdot \frac{m'n'}{mn} \\ &= \frac{1}{s} C_{d',k'}. \end{aligned}$$

□

E. Proof of Theorem 5

The following Corollary is Lemma 7 with $k = k' = \infty$ (the proof is unchanged).

Corollary 10: Let d and d' be nonnegative integers such that $(d'+1)|(d+1)$. Then

$$C_{d,\infty} \geq \frac{d'+1}{d+1} C_{d',\infty}.$$

Setting $d'=1$, Corollary 10 implies Theorem 5 for odd d. For even d, Theorem 5 immediately follows from the monotonicity $C_{d,\infty} \geq C_{d+1,\infty}$, and Theorem 5 for $d+1$ odd.

Also, when $d'=0$, Corollary 10 implies that

$$C_{d,\infty} \geq \frac{1}{d+1} \tag{17}$$

for all positive integers d, since $C_{0,\infty}=1$ is the unconstrained capacity. It is easy to check that the lower bound in (7) in Theorem 5 is tighter than that in (17), unless $d=2$ or $d=4$.

Corollary 10 ensures that a (d, ∞)-constrained binary code can be constructed from a $(1, \infty)$-constrained binary code in the manner of the previous section for odd d, by setting $s=(d+1)/2$ and $k=k'=\infty$. We next demonstrate this construction.

Let n and m be positive integers that are divisible by $\frac{d+1}{2}$, assume $n \geq m$, and let

$$n' = \frac{n}{(d+1)/2} \quad \text{and} \quad m' = \frac{m}{(d+1)/2}.$$

A (d, ∞)-constrained binary codeword on $S_{(0,0)}^{(n,m)}$ is constructed from a set of $(d+1)/2$ $(1, \infty)$-constrained binary codewords on $S_{(0,0)}^{(n',m')}$ as follows. Let $\mathcal{F}^{(n',m')}$ be a $(1, \infty)$-constrained binary code on $S_{(0,0)}^{(n',m')}$, and let

$$H = \{f_0, f_1, \cdots, f_{(d-1)/2}\}$$

be a set of $\frac{d+1}{2}$ codewords in $\mathcal{F}^{(n',m')}$. For each codeword f_t, we replace every occurrence of 0 and 1 by a $1 \times (d+1)/2$ rectangle containing the bit patterns $0^{\frac{d+1}{2}}$ and $0^t10^{\frac{d-1}{2}-t}$, respectively. Then, we merge these $(d+1)/2$ codewords into a new large codeword, on the original $m \times n$ rectangle $S_{(0,0)}^{(n,m)}$, by regarding the ith row of f_t as the $\left(\frac{(d+1)(i-1)}{2}+t+1\right)$th row of the resulting codeword for $i \in \{1, 2, \cdots, m'\}$ (the rows are ordered from bottom to top). Formally, a (d, ∞)-constrained binary codeword

$$g_H^{(n,m)}\colon S_{(0,0)}^{(n,m)} \to \{0, 1\}$$

is defined as any mapping such that the label of $S_{(x,y)}^{((d+1)/2,1)}$ under $g_H^{(n,m)}$ satisfies

$$\hat{S}_{(x,y)}^{((d+1)/2,1)} = \begin{cases} 0^{((d+1)/2)} & \text{if } f_t\left(\dfrac{x}{(d+1)/2}, \dfrac{y-t}{(d+1)/2}\right) = 0 \\ 0^t10^{((d-1)/2)-t} & \text{if } f_t\left(\dfrac{x}{(d+1)/2}, \dfrac{y-t}{(d+1)/2}\right) = 1 \end{cases}$$

for $(x, y) \equiv (0, t) \bmod ((d+1)/2)$, and for each $t = 0, 1, \cdots, (d-1)/2$ (for odd d). Fig. 12 illustrates this construction for a $(5, \infty)$-constrained binary codeword on $S_{(0,0)}^{(12,9)}$, i.e., $d=5$, $m=9$, and $n=12$.

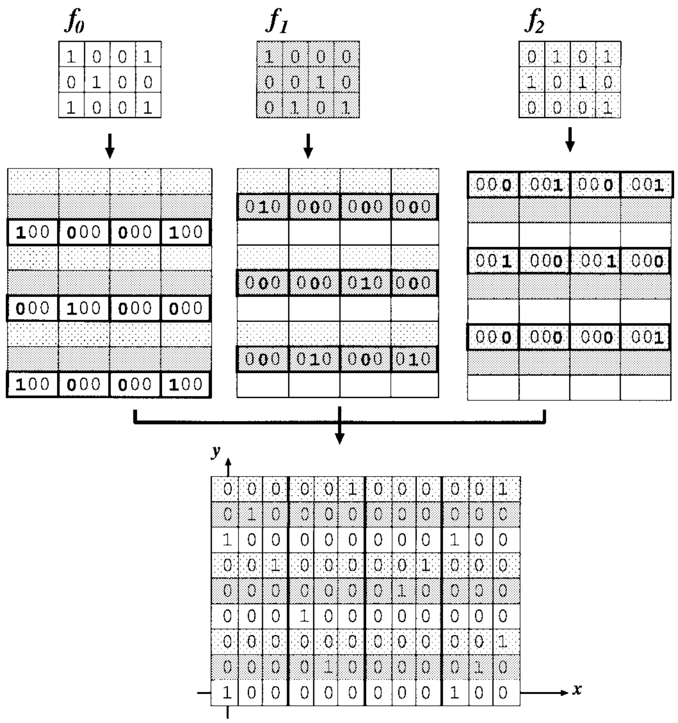

Fig. 12. A $(5, \infty)$-constrained binary codeword $g_H^{(12, 9)}$ on $S_{(0, 0)}^{(12, 9)}$ constructed from three $(1, \infty)$-constrained binary codewords f_0, f_1, f_2 on $S_{(0, 0)}^{(4, 3)}$. See Theorem 5.

F. Proof of Theorem 6

Let s be an integer such that $1 \le s \le d$, let $r = d \bmod s$, and let $\mathcal{V}_{d, s}$ be the set of all block-diagonal $(s+d) \times (s+d)$ binary matrices of the form

$$\begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \ddots & & \\ & & & A_{1+\lfloor d/s \rfloor} & \\ & & & & A' \end{bmatrix}$$

where each A_i is an $s \times s$ matrix and A' is an $r \times r$ matrix, and each row and each column of A_i and A' has at most one 1, and all the nonblock-diagonal elements are 0's. If a mapping $f\colon \boldsymbol{Z}^2 \to \{0, 1\}$ satisfies $\hat{S}_{(x, y)}^{(s+d)} \in \mathcal{V}_{d, s}$ whenever

$$x \equiv y \equiv 0 \bmod (s+d)$$

then f is a (d, ∞)-constrained codeword, since the number of 0's between consecutive 1's is at least d. Let $\mathcal{F}$ be the set of all such mappings f. Given integers n and m, we define a (d, ∞)-constrained binary code $\mathcal{F}^{(n, m)}$ on $S_{(0, 0)}^{(n, m)}$ such that each codeword in $\mathcal{F}^{(n, m)}$ is the restriction of a codeword in $\mathcal{F}$ to $S_{(0, 0)}^{(n, m)}$. Then, we have

$$C_{d, \infty} \ge \lim_{m, n \to \infty} \frac{\log_2 |\mathcal{F}^{(n, m)}|}{mn} = \frac{\log_2 |\mathcal{V}_{d, s}|}{(s+d)^2}. \qquad (18)$$

Since each A_i and A' has at most one 1 in each row and in each column, the number of choices for each A_i is (enumerating over nonzero rows and columns)

$$\sum_{i=0}^{s} \binom{s}{i}^2 i! = s! \sum_{i=0}^{s} \binom{s}{i} \frac{1}{i!}$$

and the number of choices for A' is

$$\sum_{i=0}^{r} \binom{r}{i}^2 i! = r! \sum_{i=0}^{r} \binom{r}{i} \frac{1}{i!}.$$

Hence

$$|\mathcal{V}_{d, s}| = \left(s! \sum_{i=0}^{s} \binom{s}{i} \frac{1}{i!} \right)^{1+\lfloor d/s \rfloor} \left(r! \sum_{i=0}^{r} \binom{r}{i} \frac{1}{i!} \right).$$

The maximization in (8) follows since (18) holds for every $s \le d$. $\square$

G. Proof of Corollary 2

For $d \le 13$, Corollary 2 is verified by direct calculation. Note that the right-hand side of (8) is greater than or equal to

$$\frac{\log_2 \left(d! \sum_{i=0}^{d} \binom{d}{i} \frac{1}{i!} \right)}{2d^2}$$

(by setting $s = d$ in the maximization of (8)), and Stirling's inequality [6] implies that

$$\begin{aligned} \frac{\log_2 \left(d! \sum_{i=0}^{d} \binom{d}{i} \frac{1}{i!} \right)}{2d^2} &\ge \frac{\log_2 (d!)}{2d^2} \\ &> \frac{\log_2 \left(\sqrt{2\pi d} \left(\frac{d}{e}\right)^d \right)}{2d^2} \\ &> \frac{\log_2 \left(\frac{d}{e}\right)}{2d}. \end{aligned} \qquad (19)$$

But for all $d \ge 14$

$$\frac{\log_2 \left(\frac{d}{e}\right)}{2d} \ge \frac{2C_{1, \infty}}{d} > \frac{C_{1, \infty}}{1 + ((d-1)/2)} \ge \frac{C_{1, \infty}}{1 + \lfloor d/2 \rfloor}. \quad \square$$

H. Proof of Theorem 7

Let n and m be divisible by d with $n \ge m$, and let $\mathcal{F}^{(n, m)}$ be any $(0, k)$-constrained binary code on the $n \times m$ rectangle $S_{(0, 0)}^{(n, m)}$. We break $S_{(0, 0)}^{(n, m)}$ into squares $S_{(x, y)}^{(k+1)}$ of size $(k+1) \times (k+1)$, where $x \equiv y \equiv 0 \bmod (k+1)$. Since every codeword in $\mathcal{F}^{(n, m)}$ must have at least one 1 in each row and in each column of each $\hat{S}_{(x, y)}^{(k+1)}$, we have

$$\begin{aligned} C_{0, k} &\le \lim_{m, n \to \infty} \frac{\log_2 |\mathcal{F}^{(n, m)}|}{mn} \\ &\le \frac{\log_2 \left(2^{(k+1)^2} - \sum_{i=1}^{k+1} \binom{k+1}{i} (2^{k+1} - 1)^{k+1-i} \right)}{(k+1)^2}, \end{aligned}$$

where

$$\binom{k+1}{i} (2^{k+1} - 1)^{k+1-i}$$

indicates the number of $(k+1)\times(k+1)$ binary squares consisting of i zero rows and $k-i$ nonzero rows. Equation (9) follows since

$$\begin{aligned}\sum_{i=1}^{k+1}\binom{k+1}{i}(2^{k+1}-1)^{k+1-i} &= (2^{k+1}-1)^{k+1}\sum_{i=1}^{k+1}\binom{k+1}{i}(2^{k+1}-1)^{-i}\\ &= 2^{(k+1)^2}(1-(1-2^{-(k+1)})^{k+1}). \qquad \square\end{aligned}$$

I. Proof of Theorem 8

Let n and m be divisible by d with $n \geq m$, and let $\mathcal{F}^{(n,m)}$ be any (d,∞)-constrained binary code on the $n\times m$ rectangle $S^{(n,m)}_{(0,0)}$. We break $S^{(n,m)}_{(0,0)}$ into squares $S^{(d)}_{(x,y)}$ of size $d\times d$, where $x\equiv y\equiv 0 \bmod d$. Since every codeword in $\mathcal{F}^{(n,m)}$ can have at most one 1 in each row and in each column of each $\hat{S}^{(d)}_{(x,y)}$, we have

$$\begin{aligned}C_{d,\infty} &\leq \lim_{m,n\to\infty}\frac{\log_2|\mathcal{F}^{(n,m)}|}{mn}\\ &\leq \frac{\log_2\left(\sum_{i=0}^{d}\binom{d}{i}^2 i!\right)}{d^2} = \frac{\log_2\left(d!\sum_{i=0}^{d}\binom{d}{i}\frac{1}{i!}\right)}{d^2}.\end{aligned}$$

Since

$$\sum_{i=0}^{d}\binom{d}{i}(1/i!) \leq \sum_{i=0}^{d}\binom{d}{i} = 2^d$$

Stirling's upper bound [6] implies that

$$\begin{aligned}C_{d,\infty} &\leq \frac{\log_2\left(\sqrt{2\pi d}\left(\frac{d}{e}\right)^d e^{\frac{1}{12d}}\cdot 2^d\right)}{d^2}\\ &= \frac{1}{d^2}\left(\log_2\sqrt{2\pi d} + d\log_2\left(\frac{2d}{e}\right)+\frac{1}{12d}\log_2 e\right). \qquad \square\end{aligned}$$

J. Proof of Corollary 3

The right-hand inequality in (10) is straightforward from Theorem 3. The left-hand inequality in (10) is derived from (9) as follows:

$$\begin{aligned}1-C_{0,k} &\geq \frac{-1}{k+1}\log_2(1-2^{-(k+1)})\\ &> \frac{1}{k+1}(\log_2 e)2^{-(k+1)} \quad (\text{as } k\to\infty)\\ &= \frac{\log_2 e}{(k+1)2^{k+1}}\end{aligned}$$

where we used $-\ln(1-x) = \sum_{n=1}^{\infty}(x^n/n) > x$, for all x. $\square$

K. Proof of Corollary 4

The lower bound in (8) is greater than or equal to

$$\frac{\frac{d}{s}\log_2\left(s!\sum_{i=0}^{s}\binom{s}{i}\frac{1}{i!}\right)}{(s+d)^2}. \tag{20}$$

Since $\sum_{i=0}^{s}\binom{s}{i}(1/i!) \geq 1$, the quantity in (20) is larger than

$$\begin{aligned}\frac{d}{s(s+d)^2}\log_2(s!) &> \frac{d}{s(s+d)^2}\log_2\left(\sqrt{2\pi s}\left(\frac{s}{e}\right)^s\right)\\ &> \frac{d}{(s+d)^2}\log_2\frac{s}{e}\end{aligned}$$

for all d (using Stirling's lower bound [6]). Substituting $s=\varepsilon d$ (for any $\varepsilon\in(0,1)$)

$$\frac{d}{(s+d)^2}\log_2\frac{s}{e} = \frac{1}{(1+\varepsilon)^2 d}\left(\log_2\frac{\varepsilon}{e}+\log_2 d\right).$$

Thus for any $\varepsilon\in(0,1)$, it follows that

$$\lim_{d\to\infty}\left(\frac{d}{\log_2 d}\right)\cdot C_{d,\infty} \geq \frac{1}{(1+\varepsilon)^2}$$

and therefore,

$$\lim_{d\to\infty}\left(\frac{d}{\log_2 d}\right)\cdot C_{d,\infty} \geq 1.$$

But Theorem 8 immediately implies that

$$\lim_{d\to\infty}\left(\frac{d}{\log_2 d}\right)\cdot C_{d,\infty} \leq 1. \qquad \square$$

APPENDIX
EXISTENCE OF THE TWO-DIMENSIONAL CAPACITY

Theorem 9: The two-dimensional (d,k)-capacity $C_{d,k}$ exists.

Theorem 9 is stated in [8] and [12] without a complete proof. It is a special case of a complicated proof in the preprint [7]. We give here a concise proof using an extension of [8, Lemma 4.1.7] to double sequences. Note that both the proof in [7] and our proof below essentially depend only on the two-dimensional subadditivity.

Lemma 8: Let $\{a_{m,n}\}_{m,n=1}^{\infty}$ be a double sequence of non-negative reals such that

$$a_{m_1+m_2,n} \leq a_{m_1,n}+a_{m_2,n} \tag{21}$$
$$a_{m,n_1+n_2} \leq a_{m,n_1}+a_{m,n_2}. \tag{22}$$

Then

$$\lim_{m,n\to\infty} a_{m,n}/(mn)$$

exists and equals $\inf_{m,n\geq 1}\{a_{m,n}/(mn)\}$.

Proof: It follows from (21) and (22) that

$$a_{m_1+m_2,n_1+n_2} \leq a_{m_1,n_1}+a_{m_2,n_1}+a_{m_1,n_2}+a_{m_2,n_2}. \tag{23}$$

Also, by induction we get $a_{pm,n}\leq pa_{m,n}$ and $a_{m,pn}\leq pa_{m,n}$ for all m, n, and p.

Let

$$\alpha = \inf_{m,n\geq 1} a_{m,n}/(mn).$$

By definition, $a_{m,n}/(mn)\geq\alpha$ for every $m,n\geq 1$. Fix an arbitrarily small $\varepsilon>0$. It suffices to show that $a_{m,n}/(mn)<\alpha+\varepsilon$ for all m, n sufficiently large. Since α is the largest number less than or equal to all of the $a_{m,n}/(mn)$, there exist μ and ν such that $(a_{\mu,\nu})/(\mu\nu) < \alpha + (\varepsilon/4)$. Let p and q be positive integers such that $(a_{1,1})/p < (\varepsilon/4)$, $(a_{1,1})/q < (\varepsilon/4)$, and let i and j be integers satisfying $0 \le i < \mu$ and $0 \le j < \nu$. Then

$$\begin{aligned}\frac{a_{p\mu+i,\,q\nu+j}}{(p\mu+i)(q\nu+j)} &\le \frac{a_{p\mu,\,q\nu}}{(p\mu+i)(q\nu+j)} + \frac{a_{p\mu,\,j}}{(p\mu+i)(q\nu+j)} \\ &\quad + \frac{a_{i,\,q\nu}}{(p\mu+i)(q\nu+j)} + \frac{a_{i,\,j}}{(p\mu+i)(q\nu+j)} \\ &\le \frac{a_{p\mu,\,q\nu}}{p\mu q\nu} + \frac{a_{p\mu,\,j}}{p\mu q\nu} + \frac{a_{i,\,q\nu}}{p\mu q\nu} + \frac{a_{i,\,j}}{p\mu q\nu} \\ &\le \frac{pqa_{\mu,\,\nu}}{pq\mu\nu} + \frac{pja_{\mu,\,1}}{pq\mu\nu} + \frac{iqa_{1,\,\nu}}{pq\mu\nu} + \frac{ija_{1,\,1}}{pq\mu\nu} \\ &< \left(\alpha + \frac{\varepsilon}{4}\right) + \frac{a_{1,\,1}}{q} + \frac{a_{1,\,1}}{p} + \frac{a_{1,\,1}}{pq} \\ &< \alpha + \varepsilon. \end{aligned} \tag{24}$$

For any $m \ge p\mu$ and any $n \ge q\nu$ we can always write $m = p'\mu + i$ and $n = q'\nu + j$, where $p' \ge p$, $q' \ge q$, $0 \le i < \mu$, and $0 \le j < \nu$, and thus the inequalities above show that $a_{m,n}/(mn) < \alpha + \varepsilon$ for all $m \ge p\mu$ and $n \ge q\nu$. □

Proof of Theorem 9: Let

$$a_{m,\,n} = \log_2 N_{m,\,n}^{(d,\,k)}.$$

Then, we have

$$a_{m_1+m_2,\,n} \le a_{m_1,\,n} + a_{m_2,\,n}$$

since $N_{m_1+m_2,\,n}^{(d,\,k)} \le N_{m_1,\,n}^{(d,\,k)} N_{m_2,\,n}^{(d,\,k)}$, because the right-hand side is the number of patterns on an $(m_1+m_2) \times n$ rectangle created by concatenating valid patterns on an $m_1 \times n$ rectangle and valid patterns on an $m_2 \times n$ rectangle. Similarly, we have

$$a_{m,\,n_1+n_2} \le a_{m,\,n_1} + a_{m,\,n_2}.$$

Hence, Lemma 8 can be applied to this double sequence $\{a_{m,\,n}\}_{m,\,n=1}^{\infty}$. □

Acknowledgment

This paper was motived in large part by Paul Siegel's presentation of [17] and by interesting discussions with Ron Roth at the 1997 Information Theory Workshop in Svalbard, Norway. The authors wish to thank Dan Goldstein for a helpful suggestion in Theorem 3, Brian Marcus and Bill Weeks for providing preprints [2] and [23], respectively, and Tuvi Etzion and Ron Roth for providing preprint [21].

References

[1] J. J. Ashley and P. H. Siegel, "A note on the Shannon capacity of run-length-limited codes," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 601–605, July 1987.
[2] J. J. Ashley and B. H. Marcus, "Two-dimensional lowpass filtering codes," IBM Res. Div., Almaden Res. Ctr., IBM Res. Rep. RJ 10045 (90541), Oct. 1996.
[3] N. J. Calkin and H. S. Wilf, "The number of independent sets in a grid graph," *SIAM J. Discr. Math.*, vol. 11, pp. 54–60, Feb. 1998.
[4] W. H. Erxleben and M. W. Marcellin, "Error-correcting two-dimensional modulation codes," *IEEE Trans. Inform. Theory*, vol. 41, pp. 1116–1126, July 1995.
[5] T. Etzion, "Cascading methods for runlength-limited arrays," *IEEE Trans. Inform. Theory*, vol. 43, pp. 319–324, Jan. 1997.
[6] W. Feller, *An Introduction to Probability Theory and its Applications*, vol. 1, 3rd ed. New York: Wiley, 1968, p. 54.
[7] J. Justesen and Y. M. Shtarkov, "The combinatorial entropy of images," preprint.
[8] D. Lind and B. H. Marcus, *An Introduction to Symbolic Dynamics and Coding.* New York: Cambridge Univ. Press, 1995.
[9] M. W. Marcellin and H. J. Weber, "Two-dimensional modulation codes," *IEEE J. Select. Areas Commun.*, vol. 10, pp. 254–266, Jan. 1992.
[10] B. H. Marcus, P. H. Siegel, and J. K. Wolf, "Finite-state modulation codes for data storage," *IEEE J. Select. Areas Commun.*, vol. 10, pp. 5–37, Jan. 1992.
[11] N. Markley and M. Paul, "Matrix subshifts for $\boldsymbol{Z}^{\nu}$ symbolic dynamics," *Proc. London Math. Soc.*, vol. 43, no. 3, pp. 251–272, 1981.
[12] ——, *Maximal Measures and Entropy for $\boldsymbol{Z}^{\nu}$ Subshifts of Finite Type*. (Classical Mechanics and Dynamical Systems, Dekker Notes in Mathematics, vol. 70, R. Devaney and Z. Nitecki, Eds.) New York: Marcel Dekker, 1981, pp. 135–157.
[13] S. W. McLaughlin, "The construction of M-ary (d,∞) codes that achieve capacity and have the fewest number of encoder states," *IEEE Trans. Inform. Theory*, vol. 43, pp. 699–703, Mar. 1997.
[14] S. W. McLaughlin, J. Luo, and Q. Xie, "On the capacity of M-ary runlength-limited codes," *IEEE Trans. Inform. Theory*, vol. 41, pp. 1508–1511, Sept. 1995.
[15] E. K. Orcutt and M. W. Marcellin, "Redundant multitrack (d,k) codes," *IEEE Trans. Inform. Theory*, vol. 39, pp. 1744–1750, Sept. 1993.
[16] P. N. Perry, "Runlength-limited codes for single error detection in the magnetic recording channel," *IEEE Trans. Inform. Theory*, vol. 41, pp. 809–815, May 1995.
[17] P. H. Siegel, "Problems in coding for multi-dimensional storage devices," in *IEEE Information Theory Workshop* (Longyearbyen, Norway, June 1997).
[18] P. H. Siegel and J. K. Wolf, "Bit stuffing bounds on the capacity of 2-dimensional constrained arrays," in *Proc. IEEE Int. Symp. Information Theory* (Boston, MA, Aug. 1998), p. 323.
[19] R. Talyansky, *Coding for Two-Dimensional Constraints*, M.Sc. thesis, Computer Science Dep., Technion–Israel Inst. Technol., Haifa, Israel, 1997 (in Hebrew).
[20] R. Talyansky, T. Etzion, and R. M. Roth, "Efficient code constructions for certain two-dimensional constraints," in *Proc. IEEE Int. Symp. Information Theory* (Ulm, Germany, June 1997), p. 387.
[21] ——, "Efficient code constructions for certain two-dimensional constraints," *IEEE Trans. Inform. Theory*, vol. 45, pp. 794–799, Mar. 1999.
[22] B. V. Vasic, "Capacity of channels with redundant multitrack (d,k) constraints: The $k<d$ case," *IEEE Trans. Inform. Theory*, vol. 42, pp. 1546–1548, Sept. 1996.
[23] W. Weeks, IV and R. E. Blahut, "The capacity and coding gain of certain checkerboard codes," *IEEE Trans. Inform. Theory*, vol. 44, pp. 1193–1203, May 1998.