Average Number of Facets per Cell in Tree-Structured Vector Quantizer Partitions

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ABSTRACT

Upper and lower bounds are derived for the average number of facets per cell in the encoder partition of binary Tree-Structured Vector Quantizers. The achievability of the bounds is described as well. It is shown in particular that the average number of facets per cell for unbalanced trees must lie asymptotically between 3 and 4 in $\mathcal{R}^2$, and each of these bounds can be achieved, whereas for higher dimensions it is shown that an arbitrarily large percentage of the cells can each have a linear number (in codebook size) of facets. Analogous results are also indicated for balanced trees.

SUMMARY

A binary *Tree-Structured Vector Quantizer (TSVQ)* Q can formally be defined recursively by *cutting* (or *splitting*) one cell of an existing TSVQ by a hyperplane. As in general VQ, TSVQ's also partition $\mathcal{R}^d$ into a finite set of convex polytopal cells. This follows from the fact that every encoding region is a finite intersection of half-spaces. It will be assumed throughout that the intersection of any cell-splitting hyperplane with a face of the split cell is of lower dimension then that of the face itself or equivalently that a general position restriction holds.

A *facet* of a convex polytope in $\mathcal{R}^d$ is any $(d-1)$-dimensional face of the polytope. Two cells in a quantizer partition are *neighbors* if each has a distinct facet, one of which is a subset of the other. Equivalently, two cells are neighbors if the intersection of their closures has dimension $d-1$. For a VQ encoder partition in general position, there is a one-to-one correspondence between the facets of a cell and the cell's neighbors. However, for TSVQ, it is possible that one cell could be adjacent to several other cells via the same facet; in general, the number of facets per cell is less than or equal to the number of neighbors of the cell. Often, however, these two quantities are very similar or equal. For a given convex polytopal partition Ω of $\mathcal{R}^d$ into n cells, define

1) $F_d(n)$ = average number of facets per cell in Ω.
2) $G_d(n) = nF_d(n)$
3) $M_d(n)$ = maximum number of facets of a cell in Ω.

Note that since every cell of any vector quantizer with n codevectors cannot share more than one facet with any other cell we obtain the trivial upper bound $F_d(n) \leq n-1$. In two dimensions, a straightforward application of Euler's theorem for planar graphs shows that $F_2(n) \leq 6$ (i.e. not restricted to TSVQ).

In this paper we derive several bounds on $F_d(n)$ for TSVQ and point out the achievability of these. Specifically, it is shown that for 2-dimensional unbalanced TSVQ, the average number of facets per cell is asymptotically bounded above by 4 and below by 3, and that the bounds are achievable. For higher dimensional spaces an upper bound of $n/2$ and a lower bound of 3 are given. It is also shown that $n/4$ and 3 respectively are achievable in this case. At present, it is an open question as to whether the $n/2$ bound is achievable. In $\mathcal{R}$, it is trivially always the case that $F_1(n) = 2 - 2/n$.

In $\mathcal{R}^2$, it is shown that if an asymptotically large fraction of the TSVQ cells are bounded, then $F_2(n) \approx 4$. This would lend some support to the assumption made in [1] that $F_d(n) = 2d$ for the case $d = 2$. However, for $d > 2$, this might not be the case. It is shown analogously that for balanced TSVQ with $d > 2$ the upper bound on the average number of facets per cell is reduced to $\log_2 n$. It should be emphasized, though, that the achievability of the bounds presented are best and worst cases, over the class of all TSVQ's, and it is a question for future study as to how likely they are to occur for various practical TSVQ systems.

Proposition 1 *For unbalanced TSVQ, the average number of facets per cell satisfies*

$$\begin{aligned} 3 - 4/n &\leq F_d(n) \leq n/2 - 1/2 \quad \text{for} \quad d > 2, n \geq 1 \\ 3 - 4/n &\leq F_d(n) \leq 4 - 7/n \quad \text{for} \quad d = 2, n \geq 3. \end{aligned} \tag{1}$$

The next several results exhibit the bounds' achievability.

Proposition 2 *For every $d > 2$ and $n > 1$ there exists an unbalanced TSVQ such that $F_d(n) \geq n/4$.*

Proposition 3 *For $d = 2$ and every $n > 2$ there exists an unbalanced TSVQ such that $F_d(n) = 4 - 7/n$.*

Proposition 4 *For every $d \geq 2$ and $n > 1$ there exists an unbalanced TSVQ such that $F_d(n) = 3 - 4/n$.*

The following corollary shows that there exist d-dimensional TSVQ's such that an arbitrarily large fraction of the cells each have a linear number (in codebook size) of facets.

Corollary 1 *For every $d > 2$, $n \geq 1$, and $\alpha \in (0,1)$, there exists a TSVQ with n cells such that at least αn of the cells each have at least $(1-\alpha)n$ facets.*

For balanced trees similar results are obtained, though with a reduction from linear to logarithmic bounds. The results are stated in terms of the number of cells n, in the TSVQ, though it should be remembered that balanced trees only exist when n is some integer power of 2. In the following proposition, the achievability of the lower bound for $d \geq 2$ and the upper bound for $d = 2$ are analogous to the unbalanced case. However, it is unknown at present whether the upper bound $\log_2 n$ is achievable; in fact it is unknown whether, for a fixed $d > 2$, it is possible to exhibit balanced TSVQ's such that $F_d(n)$ is unbounded.

Corollary 2 *For balanced TSVQ,*

$$\begin{aligned} 3 - 4/n &\leq F_d(n) \leq \log_2 n \quad \text{for} \quad d > 2, n > 0 \\ 3 - 4/n &\leq F_d(n) \leq 4 - 8/n \quad \text{for} \quad d = 2, n > 0. \end{aligned} \tag{2}$$

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References

[1] D. Neuhoff and D. Lee, "On the Performance of Tree-Structured Vector Quantization", *IEEE Int. Symp. Info. Theory (ISIT)*, Budapest, Hungary, June 1991.