On the Performance of Lattice Codes

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Abstract

We present a new lower bound on the probability P_e of symbol error for maximum-likelihood decoding of lattice codes on a Gaussian channel. Our bound is tight for SNR's of practical interest, as opposed to the existing bounds that are meaningful only for high SNR's. Moreover, the new lower bound on P_e is converted into an upper bound on the highest possible coding gain that may be achieved using any n-dimensional lattice code. It is shown that the effective coding gains of the densest lattice codes are much lower than the nominal coding gains, at practical symbol error rates of 10^{-5} to 10^{-7}. Furthermore, it is shown that the new bound asymptotically coincides with the Shannon limit as $n \to \infty$.

1. Introduction

Determining the maximum possible coding gain of an n-dimensional lattice code is a fundamental problem in communications. This problem has been extensively studied, for instance in [1, 2, 3] and references therein.

In [2], it is shown that, assuming high rates and high signal-to-noise ratio (SNR), the gain of a lattice code over uncoded QAM transmission can be separated into a shaping gain due to the shape of a bounding region and a coding gain due to the structure of the underlying lattice Λ. Asymptotically, as SNR $\to \infty$, the latter approaches the nominal coding gain of Λ which, in turn, depends only on the density of Λ. Thus, for very high SNR's, determining the maximum possible coding gain of an n-dimensional lattice code is equivalent to finding the densest possible lattice packing in n-dimensions.

Nevertheless, there is usually a sharp discrepancy between the nominal coding gain and the effective coding gain observed at practical signal-to-noise ratios. Hence a more careful analysis of the effective coding gain of lattice codes at practical SNR's is necessary. Such analysis is presented in this work.

2. Preliminaries

In this section, we first introduce some notation, and then establish certain well known results that will be useful later in the paper.

Let $\mathcal{S}$ be an open n-dimensional sphere of radius ρ. An infinite set Λ of vectors $\mathbf{y}_1, \mathbf{y}_2, \dots$ in $\mathbb{R}^n$ is a sphere packing if the translates $\mathbf{y}_1 + \mathcal{S}, \mathbf{y}_2 + \mathcal{S}, \dots$ are pairwise disjoint. It is a *lattice* packing, or simply a *lattice*, if the vectors $\mathbf{y}_1, \mathbf{y}_2, \dots$ form a group under addition in $\mathbb{R}^n$. Without loss of generality, it is assumed that $2\rho = d(\Lambda)$ is the minimum distance between two points of Λ. Then the density $\Delta(\Lambda)$ of Λ is the fraction of the space covered by the spheres, and the center density $\delta(\Lambda)$ is the density divided by the volume V_n of a unit sphere in $\mathbb{R}^n$. It is known [1] that

$$V_n = \frac{\pi^{n/2}}{(n/2)!} = \begin{cases} \frac{\pi^k}{k!} & n = 2k \\ \frac{2^n \pi^k k!}{n!} & n = 2k+1 \end{cases} \tag{1}$$

where $(n/2)! \stackrel{\text{def}}{=} \Gamma(\frac{n}{2}+1)$ for both odd and even n, and $\Gamma(t) = \int_0^\infty u^{t-1} e^{-u} du$ is Euler's Gamma function.

The Voronoi cell of a point $\mathbf{y} \in \Lambda$ is a convex polyhedron, which consists of all the points in $\mathbb{R}^n$ that are at least as close to $\mathbf{y}$ as to any other point in Λ. We let Π denote the Voronoi cell of the origin of $\mathbb{R}^n$. (It is easy to see that for lattice packings, Voronoi cells of all the points are congruent to each other.) The *volume* of a lattice Λ

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is defined as the volume of Π, that is $V(\Lambda) = V(\Pi)$. The asymptotic, or the *nominal*, coding gain of Λ can then be expressed as

$$\gamma(\Lambda) \;=\; 4\,\delta(\Lambda)^{2/n} \;=\; \frac{d(\Lambda)^2}{V(\Lambda)^{2/n}} \tag{2}$$

Now let $\Omega = \Lambda + \mathbf{a}$ be the translate of an n-dimensional lattice Λ by a vector $\mathbf{a}$, and let $\mathbb{D}$ be a connected, measurable, non-empty bounded region of $\mathbb{R}^n$. Then a *lattice code* $\mathbb{C} = \mathbb{C}(\Lambda, \mathbb{D})$ is defined by $\mathbb{C} = \Omega \cap \mathbb{D}$, and $\mathbb{D}$ is called the *support region* of the code. Because the support region is bounded and Λ is nowhere dense, a lattice code has finitely many points, say $\mathbb{C} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_M\}$. The quantity $R = \log_2(M)/n$ is called the *rate* of the code $\mathbb{C}$.

Given a point $\mathbf{y} \in \mathbb{R}^n$, we define the *power* of $\mathbf{y}$ as $\|\mathbf{y}\|^2/n$. The average power of the code $\mathbb{C}$ is then given by

$$P_{av} \;=\; \frac{1}{M}\sum_{i=1}^{M} \frac{\|\mathbf{y}_i\|^2}{n} \;=\; \frac{\sum_{i=1}^{M} \mathbf{y}_i \cdot \mathbf{y}_i}{nM}$$

If the number of codewords M is large, then it can be approximated as $M \simeq V(\mathbb{D})/V(\Pi)$. Thus, we have

$$R \;\simeq\; \frac{\log_2(V(\mathbb{D})/V(\Pi))}{n} \tag{3}$$

and

$$P_{av} \;\simeq\; \frac{\sum_{i=1}^{M} (\mathbf{y}_i \cdot \mathbf{y}_i)\; V(\Pi)}{nV(\mathbb{D})} \tag{4}$$

provided R is sufficiently large. The numerator of (4) is a Riemann sum that can be approximated by $\int_{\mathbb{D}} \mathbf{x} \cdot \mathbf{x}\, d\mathbf{x}$. This, along with equations (3) and (4), is known [2] as the *continuous approximation*. Using the continuous approximation, we have

$$P_{av} \;\simeq\; \mathrm{G}(\mathbb{D}) V(\mathbb{D})^{2/n} \tag{5}$$

where

$$\mathrm{G}(\mathbb{D}) \;=\; \frac{\int_{\mathbb{D}} \mathbf{x} \cdot \mathbf{x}\, d\mathbf{x}}{nV(\mathbb{D})^{\frac{2}{n}+1}}$$

is the normalized second moment of the support region $\mathbb{D}$. Notice that the average power of a lattice code $\mathbb{C} = \mathbb{C}(\Lambda, \mathbb{D})$ depends only on $\mathbb{D}$. The quantity

$$\gamma_s(\mathbb{D}) \;=\; \frac{1}{12\,\mathrm{G}(\mathbb{D})}$$

is known [2] as the shaping gain of the support region $\mathbb{D}$.

If a point $\mathbf{y} \in \mathbb{C}(\Lambda, \mathbb{D})$ is transmitted through an additive white Gaussian noise (AWGN) channel, the received point is given by $\mathbf{y} + \boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is a vector of i.i.d. Gaussian random variables with zero mean and variance σ^2. We define the normalized signal-to-noise ratio as

$$\mathrm{SNR_{norm}} \;=\; \frac{P_{av}}{(2^{2R} - 1)\sigma^2} \tag{6}$$

Since the capacity of the AWGN channel is given by

$$\frac{1}{2}\,\log_2\left(1 + \frac{P_{av}}{\sigma^2}\right)$$

Shannon's theorem [5] for Gaussian channels has an elegant statement in terms of $\mathrm{SNR_{norm}}$. Namely, arbitrarily small probabilities of symbol error can be achieved arbitrarily close to $\mathrm{SNR_{norm}} = 1 = 0\,\mathrm{dB}$.

For high rates R, we have $2^{2R} - 1 \simeq 2^{2R}$ in the denominator of (6). Further, using (3) and (5), we conclude that the normalized signal-to-noise ratio can be approximated by

$$\mathrm{SNR_{norm}} \;\simeq\; \frac{P_{av}}{2^{2R}\sigma^2} \;\simeq\; \frac{\mathrm{G}(\mathbb{D}) V(\Pi)^{2/n}}{\sigma^2}$$

Combining this with (2) gives

Lemma 1.

$$\rho/\sigma \;\simeq\; \sqrt{3\gamma_s(\mathbb{D})\;\gamma(\Lambda)\;\mathrm{SNR_{norm}}}$$

Lemma 1 is well known, see for instance [2, 6]. The approximation of Lemma 1 is accurate for high rates.

3. Error analysis

In this section, we derive a new lower bound on the probability of symbol error for maximum-likelihood decoding of n-dimensional lattice codes on an AWGN channel. The bound is not asymptotic in SNR; it is reasonably tight at SNR's of practical interest, as will be shown later in this section (see Figure 2). Moreover, as the bound applies to *any* lattice code, we have effectively bounded the performance of the *best possible* lattice codes in n dimensions.

The channel output $\mathbf{y} + \boldsymbol{\eta}$ is decoded to $\mathbf{y} \in \mathbb{C}$ under maximum-likelihood decoding, if and only if $\mathbf{y} + \boldsymbol{\eta}$ belongs to the Voronoi cell of $\mathbf{y}$ in the code $\mathbb{C} = \mathbb{C}(\Lambda, \mathbb{D})$. Thus, the probability of correct decoding is given by

$$P_c \;=\; \int_{\Pi} f(\mathbf{x})\; d\mathbf{x}. \tag{7}$$

where

$$f(\mathbf{x}) \;=\; \frac{1}{(\sqrt{2\pi}\sigma)^n}\, \exp\left(\frac{-\mathbf{x} \cdot \mathbf{x}}{2\sigma^2}\right)$$

is the probability density function of $\boldsymbol{\eta}$. (In fact, if $\mathbf{y}$ lies close to the boundary of $\mathbb{D}$, then (7) is not necessarily valid, since then the Voronoi cell of $\mathbf{y}$ in the lattice code $\mathbb{C}(\Lambda, \mathbb{D})$ is not necessarily equal to the Voronoi cell of $\mathbf{y}$ in the lattice Λ, which is congruent to Π. However, we show in [7] that for high-rate lattice codes, this boundary effect is negligible.)

Now let $\mathcal{S}(r)$ denote the n-dimensional sphere of radius r about the origin, having the same volume as Π. This sphere is sometimes called [4] the *equivalent sphere* of Π. The volume of $\mathcal{S}(r)$ is $V_n r^n$, and its radius is given by

$$r = \frac{V(\Pi)^{1/n}}{V_n^{1/n}} = \frac{V(\Lambda)^{1/n}\Gamma(\frac{n}{2}+1)^{1/n}}{\sqrt{\pi}} \quad (8)$$

in view of (1). The following simple, but key, observation dates back to the work of Shannon [5] (see also [8, p.329]), and leads to most of the results in this section.

Lemma 2.

$$\int_{\Pi} f(\mathbf{x})\, d\mathbf{x} \le \int_{\mathcal{S}(r)} f(\mathbf{x})\, d\mathbf{x} \quad (9)$$

Proof. Let $\Phi = \Pi \setminus (\Pi \cap \mathcal{S}(r))$ and $\Psi = \mathcal{S}(r) \setminus (\Pi \cap \mathcal{S}(r))$, as in Figure 1. It is obvious that (9) is equivalent to

$$\int_{\Phi} f(\mathbf{x})\, d\mathbf{x} \le \int_{\Psi} f(\mathbf{x})\, d\mathbf{x}$$

Notice that $V(\Phi) = V(\Psi)$, by the definition of the equivalent sphere $\mathcal{S}(r)$. Furthermore

$$f(\mathbf{x}) \le \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left(\frac{-r^2}{2\sigma^2}\right) \quad \text{for all } \mathbf{x} \in \Phi$$

$$f(\mathbf{x}) \ge \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left(\frac{-r^2}{2\sigma^2}\right) \quad \text{for all } \mathbf{x} \in \Psi$$

since $f(\cdot)$ is a decreasing function of the distance from the origin. Therefore

$$\int_{\Phi} f(\mathbf{x})\, d\mathbf{x} \le \frac{V(\Phi)\, e^{-r^2/2\sigma^2}}{(\sqrt{2\pi}\sigma)^n} \le \int_{\Psi} f(\mathbf{x})\, d\mathbf{x}$$

which completes the proof of the lemma. ∎

The usefulness of Lemma 2 lies in the fact that the integral on the left-hand side of (9) is often difficult to compute, whereas the integral on the right-hand side of (9) can be computed in closed form. Indeed, changing variables to spherical coordinates,

$$\begin{aligned}
x_1 &= \sigma u \cos\theta_1 \\
x_2 &= \sigma u \sin\theta_1 \cos\theta_2 \\
x_3 &= \sigma u \sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\ \vdots \\
x_{n-1} &= \sigma u \sin\theta_1 \cdots \sin\theta_{n-2} \cos\theta_{n-1} \\
x_n &= \sigma u \sin\theta_1 \cdots \sin\theta_{n-2} \sin\theta_{n-1}
\end{aligned}$$

it can be shown [7] that

$$\int_{\mathcal{S}(r)} f(\mathbf{x})\, d\mathbf{x} = \frac{n}{2^{n/2}(n/2)!} \int_0^{r/\sigma} u^{n-1} e^{-u^2/2}\, du \quad (10)$$

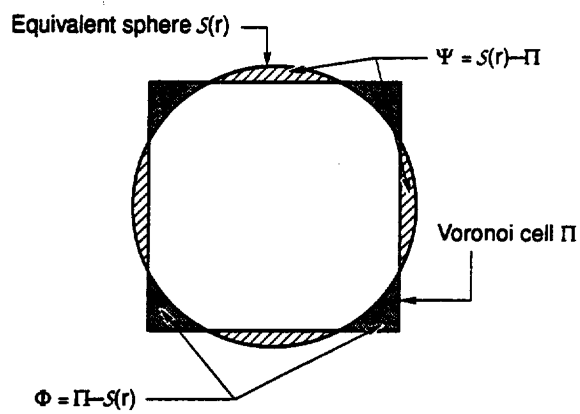

Figure 1. *The Voronoi cell and the equivalent sphere*

Let $\mathcal{I}(n)$ denote the one-dimensional integral on the right-hand side of (10). Setting $z = r^2/2\sigma^2$ and integrating by parts, we obtain

$$\mathcal{I}(n) = \mathcal{I}(n-2) - e^{-z} \frac{z^{\frac{1}{2}n-1}}{(\frac{n}{2}-1)!} \quad (11)$$

Further, it can be easily verified that $\mathcal{I}(2) = 1 - e^{-z}$ and $\mathcal{I}(1) = 1 - \text{erfc}(z^{\frac{1}{2}})$, where $\text{erfc}(\cdot)$ is the complementary error-function given by $\text{erfc}(x) = (2/\sqrt{\pi}) \int_x^{\infty} e^{-t^2} dt$. We are now ready to prove our main result in this section.

Theorem 3. *If points of an n-dimensional lattice Λ are transmitted over an AWGN channel, the probability of symbol error under maximum-likelihood decoding is lower bounded by*

$$P_e \ge e^{-z} \sum_{i=0}^{\frac{n-2}{2}} \frac{z^i}{i!} \quad (12)$$

for n even, and by

$$P_e \ge \text{erfc}(z^{\frac{1}{2}}) + e^{-z} \sum_{i=0}^{\frac{n-3}{2}} \frac{z^{i+\frac{1}{2}}}{(i+{}^1\!/_2)!} \quad (13)$$

for n odd, where

$$z = \frac{V(\Lambda)^{2/n}\Gamma(\frac{n}{2}+1)^{2/n}}{2\pi\sigma^2} = \frac{d(\Lambda)^2}{8\sigma^2\Delta(\Lambda)^{2/n}} \quad (14)$$

Proof. By Lemma 2 and (7), we have

$$P_e = 1 - P_c \ge 1 - \int_{\mathcal{S}(r)} f(\mathbf{x})\, d\mathbf{x}$$

The expressions (12) and (13) follow immediately by induction on (11). The expressions for $z = r^2/2\sigma^2$ in (14) follow from (8) and (1). ∎

For practical purposes, it is more meaningful to have a lower bound on the probability of error obtained using a lattice code $\mathbb{C} = \mathbb{C}(\Lambda, \mathbb{D})$ rather than a lattice Λ. Furthermore, it is useful to have this bound expressed in terms of the normalized signal-to-noise ratio SNR_{norm}.

Corollary 4. *If an n-dimensional lattice code* $\mathbb{C}(\Lambda, \mathbb{D})$ *is used to transmit information over an AWGN channel, then the probability of error under maximum-likelihood decoding is lower bounded by* (12) *and* (13), *with*

$$z \;=\; 6\pi^{-1}\Gamma(\tfrac{n}{2}+1)^{2/n}\gamma_s(\mathbb{D})\,\mathrm{SNR}_{\mathrm{norm}} \;+\; o(1) \qquad (15)$$

Proof. The expression for $z = r^2/2\sigma^2$ follows from (8),(2), and Lemma 1. The term $o(1)$ in (15) denotes a function of the rate R of $\mathbb{C}(\Lambda, \mathbb{D})$ that tends to zero as $R \to \infty$. ▮

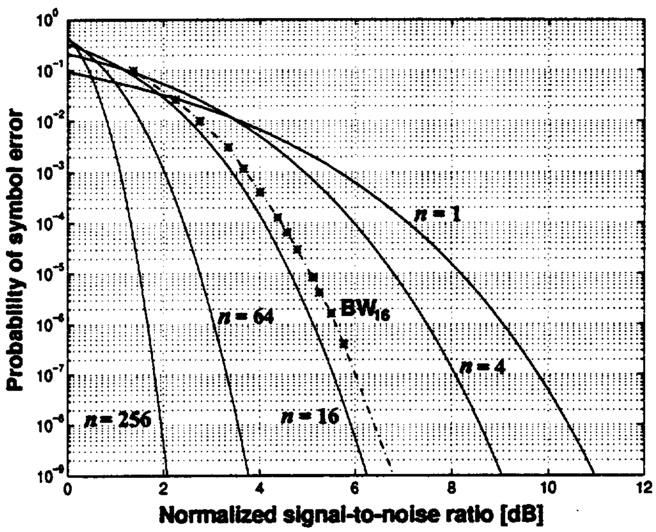

Figure 2. *Bounds on the performance of lattice codes*

Corollary 4 is a fundamental bound on the performance of any n-dimensional lattice code on an AWGN channel. This bound is plotted in Figure 2 for $n = 1, 4, 16, 64$, and 256, ignoring the term $o(1)$ in (15). Note that we assumed in Figure 2 the highest possible shaping gain for a *spherical* support region $\mathbb{D}$ in n dimensions, given by (cf. [2]):

$$\gamma_s(\mathbb{D}) \;=\; \frac{\pi(n+2)}{12\,\Gamma(\frac{n}{2}+1)^{2/n}} \qquad (16)$$

We have also included in Figure 2 the actual simulation results for a lattice code based on the 16-dimensional Barnes-Wall lattice BW_{16}, which suggest that the bound of Corollary 4 is quite tight.

4. Performance analysis

When designing a communication system for a band-limited Gaussian channel, one is generally more interested in effective coding gains than in probabilities of symbol error. In this section, we show how the lower bounds on P_e obtained in the previous section can be converted into an upper bound on the highest possible coding gain that may be achieved using an n-dimensional lattice code. For the sake of brevity, we only consider the case where n is even. A similar result for odd n will be presented elsewhere [7]. For even n, let $k = n/2$ and define

$$g_k(x) \;\stackrel{\mathrm{def}}{=}\; e^{-x}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!}\right) \qquad (17)$$

Thus (12) becomes $P_e \geq g_k(z)$. It is easy to verify that $g_k(x)$ is a continuous strictly decreasing function of x. Furthermore, $g_k(0) = 1$ and $\lim_{x\to\infty} g_k(x) = 0$, for all k.

Now let P_e denote a *fixed* desired probability of symbol error. We ask the following question: What is the minimum $\mathrm{SNR}_{\mathrm{norm}}$ that is required to achieve a probability of symbol error P_e using an n-dimensional lattice code?

From the properties of the function $g_k(x)$ in (17), it follows that the equation $g_k(x) = P_e$ has a unique solution, which we denote by z_k. We further define

$$\zeta(k; P_e) \;\stackrel{\mathrm{def}}{=}\; \frac{z_k}{k+1}$$

since assuming a spherical support region $\mathbb{D}$ as in (16), equation (15) becomes simply $z = (k+1)\mathrm{SNR}_{\mathrm{norm}} + o(1)$.

Theorem 5. *To achieve a probability of symbol error* P_e *using a lattice code* $\mathbb{C}$ *of rate* R *in* $n = 2k$ *dimensions, a normalized signal-to-noise ratio of at least*

$$\mathrm{SNR}_{\mathrm{norm}} \;\geq\; \zeta(k; P_e) + o(1) \qquad (18)$$

is required, where $o(1)$ *is a function of the rate* R *that tends to zero as* $R \to \infty$.

Proof. In view of Corollary 4, if the normalized signal-to-noise ratio does not satisfy (18), then the probability of symbol error is lower bounded by $g_k(z)$ for some $z < z_k$. Since $g_k(x)$ is a strictly decreasing function, we have $g_k(z) > g_k(z_k) = P_e$, and the theorem follows. ▮

We now consider the uncoded case, namely the case where a scaled version $c\mathbb{Z}^n$ of the integer lattice $\mathbb{Z}^n$ is used to transmit information over a Gaussian channel. (For the purpose of computing the coding gain of a lattice code $\mathbb{C}(\Lambda, \mathbb{D})$ over uncoded transmission, the scaling constant c is usually chosen in such a way that $\mathbb{C}(\Lambda, \mathbb{D})$ and $c\mathbb{Z}^n \cap \mathbb{D}$ have the same rate, assuming the continuous approximation.) It is well-known [3] that the probability of symbol error for the uncoded case can be computed exactly. Indeed, the Voronoi cell Π for the lattice $c\mathbb{Z}^n$ is a hypercube of side c, and therefore

$$\begin{aligned} P_e &= 1 - \int_{\Pi} f(\mathrm{x})\, d\mathrm{x} \\ &= 1 - \left(\frac{1}{\sqrt{2\pi}\sigma}\int_{-c/2}^{c/2} e^{-u^2/2\sigma^2}\, du\right)^n \\ &= 1 - \left(1 - \mathrm{erfc}(\rho/\sqrt{2}\sigma)\right)^n \end{aligned} \qquad (19)$$

where $\rho = c/2$ is the packing radius of the lattice $c\mathbb{Z}^n$. Observe that $\gamma(c\mathbb{Z}^n) = 1$ from (2), and hence under the continuous approximation we can write

$$\rho/\sigma \;\simeq\; \sqrt{3\gamma_s(\mathbb{D})\ \mathrm{SNR_{norm}}} \qquad (20)$$

by Lemma 1. Now let z_0 denote the unique solution of the equation $(1-\mathrm{erfc}(x))^{2k} = 1 - P_e$, and define

$$\xi(k;P_e) \;\stackrel{\mathrm{def}}{=}\; \frac{4\,z_0^2\,\Gamma(k+1)^{1/k}}{\pi(k+1)}$$

where we have again used the expression in (16) for the shaping gain $\gamma_s(\mathbb{D})$ in (20). Then, by (19) and (20), in order to achieve a probability of symbol error P_e in the uncoded case (even with spherical shaping), one needs a signal-to-noise ratio of $\mathrm{SNR_{norm}} = \xi(k;P_e)+o(1)$. Therefore, Theorem 5, implies that the ratio

$$\frac{\xi(k;P_e)}{\zeta(k;P_e)} \;=\; \frac{4\,z_0^2\,\Gamma(k+1)^{1/k}}{\pi z_k} \qquad (21)$$

is an upper bound on the coding gain that can be obtained using any high-rate lattice code in $n = 2k$ dimensions.

n	New upper bound on coding gain			Best known
	$P_e = 10^{-5}$	$P_e = 10^{-6}$	$P_e = 10^{-7}$	$P_e \to 0$
1	0	0	0	0
2	0.62	0.66	0.70	0.62
3	1.09	1.16	1.22	1.00
4	1.47	1.57	1.65	1.51
5	1.78	1.90	2.00	1.81
6	2.05	2.19	2.30	2.22
7	2.28	2.44	2.57	2.58
8	2.49	2.66	2.80	3.01
9	2.67	2.85	3.00	3.31
10	2.84	3.03	3.19	3.57
11	2.99	3.19	3.35	3.82
12	3.12	3.33	3.51	4.05
13	3.25	3.47	3.65	4.27
14	3.37	3.59	3.78	4.47
15	3.48	3.71	3.90	4.67
16	3.58	3.81	4.01	4.86
17	3.67	3.91	4.12	5.04
18	3.76	4.01	4.22	5.21
19	3.84	4.10	4.31	5.37
20	3.92	4.18	4.40	5.53
21	4.00	4.26	4.48	5.68
22	4.07	4.34	4.57	5.83
23	4.14	4.41	4.64	5.97
24	4.20	4.48	4.71	6.10
25	4.26	4.54	4.78	6.24
26	4.32	4.60	4.85	6.36
27	4.38	4.66	4.91	6.49
28	4.43	4.72	4.97	6.61
29	4.48	4.78	5.03	6.73
30	4.53	4.83	5.08	6.84
31	4.58	4.88	5.14	6.95
32	4.63	4.93	5.19	7.06

Table 1. *Upper bounds on coding gain of lattice codes*

This bound is tabulated for $P_e = 10^{-5}, 10^{-6}, 10^{-7}$ and $n = 1,2,\ldots,32$ in Table 1. All the entries in Table 1 are given in dB. Observe that the bound of (21) is not asymptotic for $P_e \to 0$; it is reasonably tight for symbol error rates of practical interest. As can be seen from Table 1, it is considerably tighter than the results obtained by computing the nominal (asymptotic for $P_e \to 0$) coding gains based on the best known [1, p.14] upper bounds on the packing density of n-dimensional lattices.

5. Asymptotic results

In this section we investigate the asymptotic behavior of the lower bound on $\mathrm{SNR_{norm}}$ of Theorem 5 as a function of dimension $n = 2k$, as $k \to \infty$. We will show that $\lim_{k\to\infty} \zeta(k;P_e) = 1$, regardless of the desired symbol error rate P_e. Thus the lower bound of Theorem 5 coincides with the Shannon limit $\mathrm{SNR_{norm}} = 0\,\mathrm{dB}$ as $k \to \infty$. This constitutes an alternative proof of the converse part of the Shannon theorem for lattice codes. Notably, our proof relies solely on the geometric notion of equivalent sphere, and does not involve information-theoretic arguments.

We start with two simple lemmas pertaining to the function $g_k(x)$ in (17). Recall that this function is strictly decreasing, and that $0 < g_k(x) \le 1$ for all $x \ge 0$.

Lemma 6. *If $x \ge k$, then*

$$g_k(x) \;\le\; \frac{e^{-x}kx^k}{k!}$$

Proof. Observe that if $x/k \ge 1$, then

$$\frac{x^k}{k!} \;\ge\; \frac{x^{k-1}}{(k-1)!} \;\ge\; \cdots \;\ge\; \frac{x^2}{2!} \;\ge\; \frac{x}{1!} \;\ge\; 1$$

Thus $g_k(x) \;=\; e^{-x}\left(1+\frac{x}{1!}+\cdots+\frac{x^{k-1}}{(k-1)!}\right) \;\le\; e^{-x}kx^k/k!$ and the lemma follows. ∎

Lemma 7. *If $0 \le x < k$, then*

$$1 - g_k(x) \;\le\; \frac{e^{-x}kx^k}{(k-x)\,k!}$$

Proof. It is easy to see that $e^x - e^x g_k(x) \;=\; \sum_{i=k}^{\infty}\frac{x^i}{i!}$. Now, for $x/k < 1$ we have

$$\sum_{i=k}^{\infty}\frac{x^i}{i!} \;\le\; \frac{x^k}{k!}\sum_{i=0}^{\infty}\frac{x^i}{k^i} \;=\; \frac{x^k}{k!}\cdot\frac{1}{1-\frac{x}{k}}$$

Hence

$$e^x\Big(1-g_k(x)\Big) \;\le\; \frac{kx^k}{(k-x)\,k!}$$

and the lemma follows. ∎

We are now ready to prove an asymptotic bound on the unique solution z_k of the equation $g_k(x) = P_e$, which holds for any fixed P_e in the interval $(0,1)$.

Lemma 8. *For any $P_e \in (0,1)$ and for any*

$$0 < \varepsilon_1 < 1 < \varepsilon_2 < \infty$$

there exists a k_0, such that for all $k \geq k_0$ we have

$$\varepsilon_1 k \;\leq\; z_k \;\leq\; \varepsilon_2 k$$

Proof. Since $g_k(x)$ is a strictly decreasing function, it suffices to show that

$$g_k(\varepsilon_2 k) \;\leq\; P_e \;\leq\; g_k(\varepsilon_1 k) \tag{22}$$

for all sufficiently large k. As $P_e \in (0,1)$ is fixed, the inequalities in (22) would follow if we knew that

$$\lim_{k\to\infty} g_k(\varepsilon_2 k) \;=\; 0 \tag{23}$$

$$\lim_{k\to\infty} g_k(\varepsilon_1 k) \;=\; 1 \tag{24}$$

We first prove the limit in (23). Since $\varepsilon_2 > 1$, the condition of Lemma 6 applies, and we have

$$g_k(\varepsilon_2 k) \;\leq\; \frac{k\,e^{-\varepsilon_2 k}(\varepsilon_2 k)^k}{k!} \;\leq\; \frac{k\,e^{-\varepsilon_2 k}\varepsilon_2^k k^k e^k}{k^k} \tag{25}$$

where the second inequality follows from the well-known fact that $k^k/e^k \leq k!$ for all $k \geq 1$. Rearranging the right-hand side of (25), we obtain

$$g_k(\varepsilon_2 k) \;\leq\; k e^{-k(\varepsilon_2 - \ln\varepsilon_2 - 1)} \tag{26}$$

Now, the function $x - \ln x - 1$ is strictly positive for $x \neq 1$, and therefore the right-hand side of (26) tends to zero as $k \to \infty$. Since $g_k(x) > 0$ for all x, this establishes (23). To prove (24), we first rewrite it as

$$\lim_{k\to\infty}\Big(1 - g_k(\varepsilon_1 k)\Big) \;=\; 0 \tag{27}$$

Note that again $1 - g_k(x) \geq 0$ for all x. Further, since $\varepsilon_1 < 1$, the condition of Lemma 7 applies, and we have

$$1 - g_k(\varepsilon_1 k) \;\leq\; \frac{k\,e^{-\varepsilon_1 k}(\varepsilon_1 k)^k}{(k - \varepsilon_1 k)\,k!} \tag{28}$$

Starting with (28) and using arguments similar to those employed in the proof of (23), it can be shown that

$$1 - g_k(\varepsilon_1 k) \;\leq\; \frac{e^{-k(\varepsilon_1 - \ln\varepsilon_1 - 1)}}{1 - \varepsilon_1} \;\;\stackrel{k\to\infty}{\longrightarrow}\;\; 0$$

This establishes (27) and (24), and hence completes the proof of the lemma. ▮

Our main result in this section implies that the bound of Thorem 5 coincides with the Shannon limit, asymptotically as $k \to \infty$.

Theorem 9. *For any $P_e \in (0,1)$,*

$$\lim_{k\to\infty} \zeta(k; P_e) \;=\; 1$$

Proof. Fix arbitrary $0 < \varepsilon_1 < 1 < \varepsilon_2 < \infty$. By Lemma 8, $z_k \geq \varepsilon_1 k$ for all sufficiently large k, and hence

$$\liminf_k \zeta(k; P_e) \;\geq\; \varepsilon_1 \lim_{k\to\infty} \frac{k}{k+1} \;=\; \varepsilon_1 \tag{29}$$

Since the value of $\varepsilon_1 < 1$ in (29) is arbitrary, it follows that $\liminf_k \zeta(k; P_e) \geq 1$. By a similar argument

$$\limsup_k \zeta(k; P_e) \;\leq\; \varepsilon_2 \lim_{k\to\infty} \frac{k}{k+1} \;=\; \varepsilon_2 \tag{30}$$

and since $\varepsilon_2 > 1$ is arbitrary, $\limsup_k \zeta(k; P_e) \leq 1$. This implies that the limit of $\zeta(k; P_e)$ exists and is equal to 1. ▮

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