Distortion Bounds and Channel Code Rates for Progressive Quantization[1]

Thomas Stockhammer
Institute for Communications Engineering
Munich University of Technology
80290 Munich, Germany
email: stockhammer@ei.tum.de

Kenneth Zeger
Department of Electrical and Computer Engineering
University of California, San Diego
La Jolla, CA 92093-0407
email: zeger@ucsd.edu

Abstract — **Progressive quantization is studied for transmission over noisy channels. For a finite set of decodable transmission rates, bounds on the minimum mean-squared distortion are derived. An asympotically optimal schedule of channel code rates is derived as a function of the transmission rate.**

I. Problem Formulation

The problem is stated for scalar random variables and mean-squared error, but it can be generalized to vector quantizers and pth power distortions. Let X be a scalar random variable and let $0 \le b_1 \le b_2 \le \ldots \le b_L$ be a set of *decodable transmission rates*. A *progressive quantizer* is a sequence of quantizers $Q^{(1)}, Q^{(2)}, \ldots, Q^{(L)}$ such that for each $m \in \{1, \ldots, L-1\}$, the quantizer $Q^{(m)}$ operates at rate b_i, and the quantizer encoder $Q_c^{(m)} : \mathbb{R} \to \{0,1\}^{b_m}$ has the property that for every $x \in \mathbb{R}$, the binary channel word $Q_c^{(m)}(x)$ is a prefix of the binary word $Q_c^{(m+1)}(x)$. For a given x, the sequence of quantizations $Q^{(1)}(x), Q^{(2)}(x), \ldots, Q^{(L)}(x)$ is designed to become closer to x as the transmission rate increases from b_1 towards b_L.

Suppose the source X is quantized at the maximum transmission rate b_L, and that $I = I_1 I_2 \cdots I_L$ is the resulting binary word of length b_L, where $Q_c^{(m)}(X) = I_1 \cdots I_m$, for each m. Then I_m is the additional binary string appended to $Q_c^{(m-1)}(X)$ when increasing the quantizer resolution by one stage.

Suppose each I_m is expanded to a word $\hat{I}_m$ by a channel code of rate $r_m \in [0,1]$ and then transmitted across a noisy channel and received as the binary word $\hat{J}_m$. The received word is decoded into a b_m bit vector index J_m from which a reconstruction vector $y_{J_m}^{(m)}$ is used to approximate X. That is, for each m there is a separate codebook $\{y_0^{(m)}, \ldots, y_{2^{b_m}-1}^{(m)}\}$.

Let $S = \min\{m : J_m \ne I_m\} - 1$. The random variable S is the index specifying the last stage for which the quantizer index I_m was correctly decoded after transmission. Assume the decoder knows the value of S (using error detection) and uses $Q^{(S)}(X)$ as its estimate of X. The average distortion is measured by

$$D = E[(X - y_{I_1 \ldots I_S}^{(S)})^2]$$

where the expectation is taken over both the source and channel statistics.

To understand the behavior of progressive quantization over noisy channels, we use high resolution assumptions to derive an expression for the minimum D as a function of the transmission rate, and we also obtain bounds on the channel code rates $\tilde{r}_1, \ldots, \tilde{r}_L$ that minimize D.

Let $\omega_1, \ldots, \omega_L$ be a fixed set of nonnegative numbers such that $\omega_1 + \cdots + \omega_L = 1$. Let R be the total transmission rate (assuming $R \to \infty$) and assume the decodable transmission rates are $b_m = R\sum_{i=1}^m \omega_i r_i$ for each m. Assume the channel is binary symmetric with crossover probability ϵ and that optimal channel codes are used.

Assuming maximum likelihood decoding the block error probability at the channel decoder satisfies $P_e = 2^{-nE(r)+o(n)}$ with n the channel code block length and $E(r)$ the error exponent function. Upper bounds $\overline{E}(r)$ and lower bounds $\underline{E}(r)$ on the error exponent are known [1].

High resolution implies that for each $m = 1, \ldots, L$, and with no channel noise,

$$E[(X - Q^{(m)}(X))^2] \quad = \quad 2^{-2R\sum_{i=1}^m \omega_i r_i + O(1)}.$$

[1]This work was supported in part by the National Science Foundation.

II. Channel Coder Rates and Distortion Bounds

Under certain reasonable assumptions on the quantizer cell diameters and with randomized index assignments, it is shown that

$$D \;\le\; 2^{-pR\sum_{m=1}^L \omega_m r_m + O(1)} + \sum_{m=1}^{L} 2^{-R\left(\omega_m \underline{E}(r_m) + 2\sum_{i=1}^{m-1}\omega_i r_i\right) + O(1)}$$

$$D \;\ge\; 2^{-2R\sum_{m=1}^L \omega_m r_m + O(1)} + \sum_{m=1}^{L} 2^{-R\left(\omega_m \overline{E}(r_m) + 2\sum_{i=1}^{m-1}\omega_i r_i\right) + O(1)}.$$

The average distortion can be written as

$$D \;=\; 2^{-2R\sum_{m=1}^L \omega_m r_m + O(1)} + \sum_{m=1}^{L} 2^{-R\left(\omega_m E(r_m) + 2\sum_{i=1}^{m-1}\omega_i r_i\right) + O(1)}.$$

The distortion is minimized by the channel code rates $\tilde{r}_1, \ldots, \tilde{r}_L$ which satisfy the equations:

$$E(\tilde{r}_l) \quad = \quad \frac{2}{\omega_m}\sum_{i=m}^{L}\omega_i \tilde{r}_i, \quad \text{for} \quad m = 1, \ldots, L$$

and the minimum distortion $D_{\min}$ can therefore be written as

$$D_{\min} \quad = \quad 2^{-2R\sum_{m=1}^L \omega_m \tilde{r}_m + O(1)}.$$

If the decodable transmission rates are at equally spaced intervals then $\omega_m = 1/L$ for all m and thus

$$D_{\min} \quad = \quad 2^{-(2R/L)\sum_{m=1}^L \tilde{r}_m + O(1)}.$$

Let $\delta = 2\sqrt{\epsilon(1-\epsilon)}$. A rather long derivation shows that $\sum_{m=1}^L \tilde{r}_m \to (1/4)\log_2(1/\delta)$ as $L \to \infty$. Thus the minimum distortion for $R \to \infty$ and then $L \to \infty$ is

$$D_{\min} \quad \approx \quad 2^{-(R/(2L))\log_2(1/\delta)} = \delta^{R/(2L)} \approx (4\epsilon)^{R/(4L)}.$$

A factor of $(1/(4L))\log_2(1/\delta)$ in the exponent can be thought of as a distortion penalty paid for using L levels of progressivity.

References

[1] R.G. Gallager, *Information Theory and Reliable Communication,* Wiley, New York, 1968.