# Capacity Bounds for the 3-dimensional $(0,1)$ Runlength Limited Channel

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**Abstract.** The capacity $C_{0,1}^{(3)}$ of a 3-dimensional $(0,1)$ runlength constrained channel is shown to satisfy $0.522501741838 \leq C_{0,1}^{(3)} \leq 0.526880847825$ .

## 1 Introduction

A binary sequence satisfies a 1-dimensional $(d,k)$ runlength constraint if there are at most $k$ zeros in a row, and between every two consecutive ones there are at least $d$ zeros. An $n$-dimensional binary array is said to satisfy a $(d,k)$ runlength constraint, if it satisfies the 1-dimensional $(d,k)$ runlength constraint along every direction parallel to a coordinate axis. Such an array is called *valid*. The number of valid $n$-dimensional arrays of size $m_1 \times m_2 \times \ldots \times m_n$ is denoted by $N_{m_1,m_2,\ldots,m_n}^{(d,k)}$ and the corresponding *capacity* is defined as

$$C_{d,k}^{(n)} = \lim_{m_1,m_2,\ldots m_n \to \infty} \frac{\log_2 N_{m_1,m_2,\ldots m_n}^{(d,k)}}{m_1 m_2 \cdots m_n}.$$

By exchanging the roles of 0 and 1 it can be seen that $C_{0,1}^{(n)} = C_{1,\infty}^{(n)}$ for all $n \geq 1$. A simple proof of the existence of the 2-dimensional $(d,k)$ capacities can be found in [1], and the proof can be generalized to $n$-dimensions.

It is known (e.g. see [2]) that the 1-dimensional $(0,1)$-constrained capacity is the logarithm of the golden ratio, i.e.

$$C_{0,1}^{(1)} = \log_2 \frac{1+\sqrt{5}}{2} = 0.694242\ldots$$

and in [3] very close upper and lower bounds were given for the 2-dimensional $(0,1)$-constrained capacity. The bounds in [3] were calculated with greater precision in [4] and are further slightly improved here by us (see Remark section at end for more details), now agreeing in 9 decimal positions:

$$0.587891161775 \leq C_{0,1}^{(2)} \leq 0.587891161868 \quad . \tag{1}$$

A lower bound of $C_{0,1}^{(2)} \geq 0.5831$ was obtained in [5] by using an implementable encoding procedure known as "bit-stuffing". The known bounds on $C_{0,1}^{(2)}$ have played a useful role in [1] for obtaining bounds on other $(d, k)$-constraints in two dimensions. The 3-dimensional $(0, 1)$-constrained bounds given in the present paper can play a similar role for obtaining different 3-dimensional bounds, and are also of theoretical interest. In fact, a recent tutorial paper [6] discusses an interesting connection between run length constrained capacities in more than one dimension and crossword puzzles (based on work of Shannon from 1948). In the present paper we consider the 3-dimensional $(0, 1)$ constraint, and by extending ideas from [3] our main result is to derive (in Sections 2 and 3) the following bounds on the 3-dimensional $(0, 1)$ capacity.

**Theorem 1**

$$0.522501741838 \leq C_{0,1}^{(3)} \leq 0.526880847825$$

It is assumed henceforth in this paper that $d = 0$ and $k = 1$. Two valid $m_1 \times m_2$ rectangles can be put next to each other in 3 dimensions without violating the 3-dimensional $(0, 1)$ constraint if they have no two zeros in the same positions. Define a *transfer matrix* $T_{m_1,m_2}$ to be an $N_{m_1,m_2}^{(0,1)} \times N_{m_1,m_2}^{(0,1)}$ binary matrix, such that the rows and columns are indexed by the valid 2-dimensional $m_1 \times m_2$ patterns, and an entry of $T_{m_1,m_2}$ is 1 if and only if the corresponding two rectangles can be placed next to each other in 3 dimensions without violating the $(0, 1)$ constraint. Then,

$$N_{m_1,m_2,m_3}^{(0,1)} = \mathbf{1}' \cdot T_{m_1,m_2}^{m_3-1}\mathbf{1} = \mathbf{1}' \cdot T_{m_1,m_3}^{m_2-1}\mathbf{1} = \mathbf{1}' \cdot T_{m_2,m_3}^{m_1-1}\mathbf{1}$$

where $\mathbf{1}$ is the all ones column vector and prime denotes transpose. The matrix $T_{m_1,m_2}$ meets the conditions of the Perron-Frobenius theorem [7], since it has nonnegative real elements and is irreducible (since the all one's rectangle can be placed next to any valid rectangle without violating the $(0, 1)$ constraint). Therefore the largest magnitude eigenvalue $\Lambda_{m_1,m_2}$, of $T_{m_1,m_2}$, is positive, real, and has multiplicity one. This implies that

$$\lim_{m_3 \to \infty} (N_{m_1,m_2,m_3}^{(0,1)})^{1/m_3} = \Lambda_{m_1,m_2},$$

and

$$\begin{aligned}
C_{0,1}^{(3)} &= \lim_{m_1,m_2,m_3 \to \infty} \frac{\log_2 N_{m_1,m_2,m_3}^{(0,1)}}{m_1 m_2 m_3} \\
&= \lim_{m_1,m_2 \to \infty} \frac{\log_2 \lim_{m_3 \to \infty} (N_{m_1,m_2,m_3}^{(0,1)})^{1/m_3}}{m_1 m_2} \\
&= \lim_{m_1,m_2 \to \infty} \frac{\log_2 \Lambda_{m_1,m_2}}{m_1 m_2} \\
&= \lim_{m_1 \to \infty} \frac{\log_2 \lim_{m_2 \to \infty} \Lambda_{m_1,m_2}^{1/m_2}}{m_1} \\
&= \lim_{m_1 \to \infty} \frac{\log_2 \Lambda_{m_1}}{m_1},
\end{aligned} \tag{2}$$

where $\Lambda_{m_1} = \lim_{m_2 \to \infty} \Lambda_{m_1,m_2}^{1/m_2}$. The quantities $\frac{\log_2 \Lambda_{m_1,m_2}}{m_1 m_2}$ and $\frac{\log_2 \Lambda_{m_1}}{m_1}$ can be viewed as capacities corresponding to 3-dimensional arrays with two fixed sides (lengths $m_1$ and $m_2$), and one fixed side (length $m_1$), respectively.

Upper and lower bounds on the 3-dimensional capacity can be computed directly from the inequalities (similar to the 2-dimensional case, as noted in [4])

$$\frac{\log_2 \Lambda_{m_1,m_2}}{(m_1+1)(m_2+1)} \leq C_{0,1}^{(3)} \leq \frac{\log_2 \Lambda_{m_1,m_2}}{m_1 m_2}$$

but these do not yield particularly tight bounds for values of $m_1$ and $m_2$ that result in reasonable space and time complexities (e.g. Table 1 shows that the eigenvalues $\Lambda_{m_1,m_2}$ correspond to matrices with more than 40 million elements when roughly $m_1 m_2 \geq 20$). The upper and lower capacity bounds derived in this paper agree to within $\pm 0.002$ and were computed using less than 100 Mbytes of computer memory.

## 2 Lower bound on $C_{0,1}^{(3)}$

To derive a lower bound on $C_{0,1}^{(3)}$ we generalize a method of Calkin and Wilf [3]. Since $T_{m_1,m_2}$ is a symmetric matrix, the Courant-Fischer Minimax Theorem [8, pg. 394] implies that

$$\Lambda_{m_1,m_2}^p \geq \frac{\mathbf{x}' \cdot T_{m_1,m_2}^p \mathbf{x}}{\mathbf{x}' \cdot \mathbf{x}} \tag{3}$$

for any nonzero vector $\mathbf{x}$ and any integer $p \geq 0$. Choosing $\mathbf{x} = T_{m_1,m_2}^q \mathbf{1}$ for any integer $q \geq 0$ gives

$$\Lambda_{m_1,m_2}^p \geq \frac{\mathbf{1}' \cdot T_{m_1,m_2}^{p+2q} \mathbf{1}}{\mathbf{1}' \cdot T_{m_1,m_2}^{2q} \mathbf{1}} = \frac{\mathbf{1}' \cdot T_{m_1,p+2q+1}^{m_2-1} \mathbf{1}}{\mathbf{1}' \cdot T_{m_1,2q+1}^{m_2-1} \mathbf{1}}. \tag{4}$$

Thus,

$$\begin{aligned} 2^{pC_{0,1}^{(3)}} &= \left( \lim_{m_1,m_2 \to \infty} \Lambda_{m_1,m_2}^{1/(m_1 m_2)} \right)^p = \lim_{m_1 \to \infty} \left( \lim_{m_2 \to \infty} \Lambda_{m_1,m_2}^{p/m_2} \right)^{1/m_1} \\ &\geq \lim_{m_1 \to \infty} \left( \frac{\Lambda_{m_1,p+2q+1}}{\Lambda_{m_1,2q+1}} \right)^{1/m_1} = \frac{\lim_{m_1 \to \infty} \Lambda_{m_1,p+2q+1}^{1/m_1}}{\lim_{m_1 \to \infty} \Lambda_{m_1,2q+1}^{1/m_1}} = \frac{\Lambda_{p+2q+1}}{\Lambda_{2q+1}} \end{aligned} \tag{5}$$

and therefore for any odd integer $r \geq 1$ and any integer $z > r$,

$$C_{0,1}^{(3)} \geq \frac{1}{z-r} \log_2 \left( \frac{\Lambda_z}{\Lambda_r} \right). \tag{6}$$

This lower bound on $C_{0,1}^{(3)}$ is analogous to a 2-dimensional bound in [3], but $\Lambda_z$ and $\Lambda_r$ are not eigenvalues associated with transfer matrices of 2-dimensional arrays here, and cannot easily be computed as in the 2-dimensional case. Instead, we obtain a lower bound on $\Lambda_z$ and an upper bound on $\Lambda_r$. ¿From (4) and (5) a lower bound on $\Lambda_z$ is

$$\Lambda_z = \lim_{m_2 \to \infty} \Lambda_{z,m_2}^{1/m_2} \geq \lim_{m_2 \to \infty} \left( \frac{\mathbf{1}' \cdot T_{z,v}^{m_2-1} \mathbf{1}}{\mathbf{1}' \cdot T_{z,u}^{m_2-1} \mathbf{1}} \right)^{1/((v-u)m_2)} = \left( \frac{\Lambda_{z,v}}{\Lambda_{z,u}} \right)^{1/(v-u)},$$

where $u$ is an arbitrary positive odd integer, $v > u$, and $\Lambda_{z,v}$ and $\Lambda_{z,u}$ are the largest eigenvalues of the transfer matrices $T_{z,v}$ and $T_{z,u}$, respectively.

To find an upper bound on the quantity $\Lambda_r$ for a given $r$, we apply a modified version of a method in [3]. We say that a binary matrix satisfies the $(0,1)$ *cylindrical constraint* if it satisfies the usual 2-dimensional $(0,1)$ constraint after joining its leftmost column to its rightmost column (i.e. the left and right columns can be put next to each other without violating the $(0,1)$ constraint). A binary matrix satisfies the $(0,1)$ *toroidal constraint* if it satisfies the usual 2-dimensional $(0,1)$ constraint after both joining its leftmost column to its rightmost column, and its top row to its bottom row.

**Proposition 1** *Let $s$ be a positive even integer and let $T_{m_1,m_2}$ be the transfer matrix whose rows and columns are indexed by all $(0,1)$-constrained $m_1 \times m_2$ rectangles. Let $B_{m_1,s}$ denote the transfer matrix whose rows and columns are indexed by all cylindrically $(0,1)$-constrained $m_1 \times s$ rectangles. Then,*

$$\mathrm{Trace}[T^s_{m_1,m_2}] = \mathbf{1}' \cdot B^{m_2-1}_{m_1,s}\mathbf{1}.$$

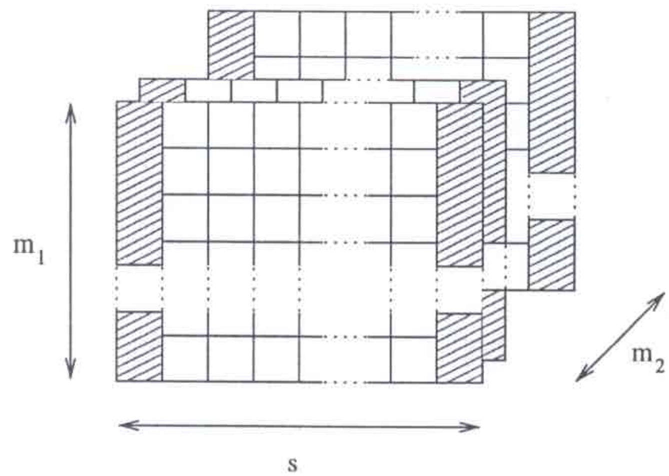


**Fig. 1.** Cylindrically $(0,1)$-constrained $m_1 \times s$ rectangles used to build cylindric $m_1 \times m_2 \times s$ arrays

For every positive integer $m_1$ and $m_2$, and every even positive integer $s$, the matrix $T^s_{m_1,m_2}$ has nonnegative eigenvalues and thus any one of its eigenvalues is upper bounded by its trace. Hence,

$$\Lambda_{m_1,m_2} \leq \mathrm{Trace}\left[T^s_{m_1,m_2}\right]^{1/s} = \left(\mathbf{1}' \cdot B^{m_2-1}_{m_1,s}\mathbf{1}\right)^{1/s} \tag{7}$$

which gives the following upper bound on $\Lambda_r$:

$$\Lambda_r = \lim_{m_2\to\infty} \Lambda^{1/m_2}_{r,m_2} \leq \lim_{m_2\to\infty} \left(\mathbf{1}' \cdot B^{m_2-1}_{r,s}\mathbf{1}\right)^{\frac{1}{sm_2}} = \xi^{1/s}_{r,s}, \tag{8}$$

where $\xi_{r,s}$ is the largest eigenvalue of $B_{r,s}$ (note that $B_{r,s}$ satisfies the Perron-Frobenius theorem for the same reasons as for $T_{m_1,m_2}$ in Section 1).

The lower bound on $C_{0,1}^{(3)}$ in (6) can now be written as

$$C_{0,1}^{(3)} \geq \frac{1}{z-r} \log_2 \left( \frac{\left(\frac{\Lambda_{z,v}}{\Lambda_{z,u}}\right)^{1/(v-u)}}{\xi_{r,s}^{1/s}} \right) \qquad \begin{matrix} r \text{ and } u \text{ odd, } s \text{ even} \\ z > r \geq 1 \\ v > u \geq 1 \\ s \geq 2 \end{matrix} \tag{9}$$

To obtain the best possible lower bound, the right hand side of (9) should be maximized over all acceptable choices of $r$, $z$, $u$, $v$, and $s$, subject to the numerical computability of the quantities $\Lambda_{z,v}$, $\Lambda_{z,u}$, and $\xi_{r,s}$. Table 1 shows the largest eigenvalues of various transfer matrices which were numerically computable. ¿From this table, the best parameters we could find for the lower bound in (9) on the capacity were $r = 3$, $z = 4$, $u = 5$, $v = 6$, and $s = 10$, yielding

$$C_{0,1}^{(3)} \geq \frac{1}{4-3} \log_2 \frac{\frac{9346.35893701}{2102.73425568}}{(80481.0598379)^{1/10}} \geq 0.522501741838.$$

## 3 Upper bound on $C_{0,1}^{(3)}$

**Proposition 2** *Let $s_1$ and $s_2$ be positive even integers and let $B^*_{s_1,s_2}$ denote the transfer matrix whose rows and columns are indexed by all toroidally $(0,1)$-constrained $s_1 \times s_2$ rectangles. If $\xi^*_{s_1,s_2}$ is the largest eigenvalue of $B^*_{s_1,s_2}$, then $C_{0,1}^{(3)} \leq \frac{1}{s_1 s_2} \log_2 \xi^*_{s_1,s_2}$.*

Note that $B_{2,s_2} = B^*_{2,s_2}$ and thus $\xi_{2,s_2} = \xi^*_{2,s_2}$. The best parameters we were able to find (from Table 1) were $s_1 = 4$ and $s_2 = 6$, and the resulting eigenvalue gave the following upper bound:

$$C_{0,1}^{(3)} \leq \frac{1}{24} \log_2 6405.69924332 \leq 0.526880847825.$$

## 4 Remark

Direct computation of eigenvalues using standard linear algebra algorithms generally requires the storage of an entire matrix. This severely restricts the matrix sizes allowable, due to memory constraints on computers. By exploiting the fact that our matrices are all binary, symmetric, and easily computable, we were able to obtain the largest eigenvalues of much larger matrices. Specifically, the eigenvalues used to obtain the capacity bounds in Theorem 1 were computed using the "power method" [8, pg. 406]. Similarly, we obtained the upper bound in (1) with the power method (computing $\Lambda_{1,21}$, $\Lambda_{1,23}$, and $\xi_{1,24}$). Originally these bounds were computed in [3] as $0.587891161 \leq C_{0,1}^{(2)} \leq 0.588339078$ (computing $\Lambda_{1,13}$, $\Lambda_{1,15}$, and $\xi_{1,6}$) and were later improved in [4] (computing $\Lambda_{1,13}$, $\Lambda_{1,14}$, and $\xi_{1,14}$) to $0.587891161775 \leq C_{0,1}^{(2)} \leq 0.587891494943$. The lower bound in (1) is from [4].

| $a$ | $b$ | $\Lambda_{a,b}$ | rows of $T_{a,b}$ | $\xi_{a,b}$ | rows of $B_{a,b}$ | $\xi^*_{a,b}$ | rows of $B^*_{a,b}$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1.61803398875 | 2 | | | | |
| | 2 | 2.41421356237 | 3 | 2.41421356237 | 3 | | |
| | 3 | 3.63138126040 | 5 | | | | |
| | 4 | 5.45770539597 | 8 | 5.15632517466 | 7 | | |
| | 5 | 8.20325919376 | 13 | | | | |
| | 6 | 12.3298822153 | 21 | 11.5517095660 | 18 | | |
| | 7 | 18.5324073775 | 34 | | | | |
| | 8 | 27.8550990963 | 55 | 26.0579860919 | 47 | | |
| | 9 | 41.8675533183 | 89 | | | | |
| | 10 | 62.9289457252 | 144 | 58.8519350815 | 123 | | |
| | 11 | 94.5852312050 | 233 | | | | |
| | 12 | 142.166150393 | 377 | 132.947794048 | 322 | | |
| | 13 | 213.682559741 | 610 | | | | |
| | 14 | 321.175161677 | 987 | 300.345852027 | 843 | | |
| | 15 | 482.741710897 | 1597 | | | | |
| | 16 | 725.584002895 | 2584 | 678.525669346 | 2207 | | |
| | 17 | 1090.58764423 | 4181 | | | | |
| | 18 | 1639.20566742 | 6765 | 1532.89283597 | 5778 | | |
| | 19 | 2463.80493521 | 10946 | | | | |
| | 20 | 3703.21728345 | 17711 | 3463.03987027 | 15127 | | |
| | 21 | 5566.11363689 | 28657 | | | | |
| | 22 | 8366.13642876 | 46368 | 7823.53857819 | 39603 | | |
| | 23 | 12574.7053170 | 75025 | | | | |
| | 24 | 18900.3867144 | 121393 | 17674.5747630 | 103682 | | |
| 2 | 2 | 5.15632517466 | 7 | 5.15632517466 | 7 | 5.15632517466 | 7 |
| | 3 | 11.1103016575 | 17 | | | | |
| | 4 | 23.9250625386 | 41 | 21.9287654025 | 35 | 21.9287654025 | 35 |
| | 5 | 51.5229210280 | 99 | | | | |
| | 6 | 110.954925971 | 239 | 100.236549238 | 199 | 100.236549239 | 199 |
| | 7 | 238.942175857 | 577 | | | | |
| | 8 | 514.563569622 | 1393 | 463.203410887 | 1155 | 463.203410887 | 1155 |
| | 9 | 1108.11608218 | 3363 | | | | |
| | 10 | 2386.33538059 | 8119 | 2146.04060032 | 6727 | 2146.04060032 | 6727 |
| | 11 | 5138.98917320 | 19601 | | | | |
| | 12 | 11066.8474924 | 47312 | 9949.63685703 | 39203 | 9949.63685703 | 39203 |
| 3 | 3 | 34.4037405361 | 63 | | | | |
| | 4 | 106.439377528 | 227 | 94.2548937790 | 181 | | |
| | 5 | 329.331697608 | 827 | | | | |
| | 6 | 1018.97101980 | 2999 | 884.498791440 | 2309 | | |
| | 7 | 3152.75734322 | 10897 | | | | |
| | 8 | 9754.81971205 | 39561 | 8421.60680806 | 30277 | | |
| | 9 | 30181.9963196 | 143677 | | | | |
| | 10 | 93384.9044989 | 521721 | 80481.0598378 | 398857 | | |
| 4 | 4 | 473.069084944 | 1234 | 404.943621498 | 933 | 355.525781764 | 743 |
| | 5 | 2102.73425567 | 6743 | | | | |
| | 6 | 9346.35893702 | 36787 | 7799.87080772 | 26660 | 6405.69924332 | 18995 |

**Table 1.** Largest eigenvalues of $T_{a,b}$, $B_{a,b}$, and $B^*_{a,b}$ are $\Lambda_{a,b}$, $\xi_{a,b}$, and $\xi^*_{a,b}$.

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