Capacity Bounds for the Hard-Triangle Model

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I. Introduction

Two-dimensional constraints play a role in optical storage devices. In particular, run length constraints in one and higher dimensions have been a subject of intense research. In two dimensions, such constraints have primarily been studied for rectangular and hexagonal lattices. We examine such constraints for an equilateral triangular non-lattice tiling of the two-dimensional plane.

A binary sequence satisfies a *one-dimensional* (d,k) *run length constraint* if there are at most k zeros in a row, and between every two consecutive ones there are at least d zeros. A two-dimensional binary rectangular array is said to satisfy a *two-dimensional* (d,k) *run length constraint*, if it satisfies the one-dimensional (d,k) run length constraint along the directions parallel to the coordinate axes. Such an array is called (d,k)*-valid*. The number of (d,k)-valid two-dimensional arrays of size $m \times n$ is denoted by $\nu_{d,k}(m,n)$ and the corresponding *capacity* is defined as $C_{d,k}^{(2)} = \lim_{m,n\to\infty} \frac{\log_2 \nu_{d,k}(m,n)}{mn}$.

Here, we consider a non-lattice tiling of the two-dimensional plane by equilateral triangles and use the center of each triangle to store a bit. Analogous to the square and hexagonal cases, we study a "hard triangle" constraint on the triangular tiling. Every triangle with a 1 in it must have all three of its neighboring triangles have 0s in them. We analyze the capacity by deriving an upper bound analytically and obtain a lower bound by exhibiting a bit stuffing algorithm for encoding arbitrary input binary sequences into the triangular tiling without violating the constraint.

II. Definitions

For any $i,j \in \mathbf{Z}$, let $[i,j] = i(\sqrt{3},0) + j\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$, $[i,j]' = [i,j] + (0,1)$, $T_1 = \{[i,j] : i,j \in \mathbf{Z}\}$, $T_2 = T_1 + (0,1)$, $T = T_1 \cup T_2$. The notation $[i,j]$ represents a point in T_1 with respect to the basis $\left\{(\sqrt{3},0),\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)\right\}$.We say that two points (or their corresponding triangles) in T are *neighbors* if the distance between them is 1.

For any $S \subset T$, a function $f : S \longrightarrow \{0,1\}$ is called a *labeling* of S. A labeling f of S satisfies the *hard-triangle constraint* if for every $t \in S$, the three nearest neighbors of t are labeled with 0s whenever $f(t) = 1$. In what follows, we will call a labeling of S *valid* if it satisfies the hard-triangle constraint, and will denote the set of valid labelings by $L(S)$. The capacity of the hard-triangle constraint will be defined analogously to the hard-square constrained capacity.

For $X,Y \in \mathbf{Z}^+$, define the array $A_{X,Y} \subset T$ as

$$\begin{aligned} A_{X,Y} &= \{[i,j] : 0 \le i \le X, 0 \le j \le Y\} \\ &\quad \cup \left\{[i,j]' : 0 \le i \le X, -1 \le j \le Y-1\right\} \end{aligned}$$

and let $\nu(X,Y)$ denote the number of valid labelings of $A_{X,Y}$. The capacity C_T corresponding to the hard-triangle constraint is defined as

$$C_T = \lim_{X,Y\to\infty} \frac{\log_2 \nu(X,Y)}{|A_{X,Y}|} = \lim_{X,Y\to\infty} \frac{\log_2 \nu(X,Y)}{2(X+1)(Y+1)}$$

This research was supported in part by the National Science Foundation and the UCSD Center for Wireless Communications.

where the right hand side follows since the size of $A_{X,Y}$ is $2(X+1)(Y+1)$.

III. Hard-triangle capacity upper bound

Let $f_1, f_2, \ldots, f_{\nu(0,Y)}$ denote the valid labelings of $A_{0,Y}$. Define the *transfer matrix* M_Y to be a $\nu(0,Y) \times \nu(0,Y)$ binary matrix, such that the rows and columns of M_Y are indexed by the valid labelings of $A_{0,Y}$, and the $(i,j)^{th}$ entry of M_Y is 1 if and only if the labeling of $A_{0,Y}$ and $A_{0,Y} + [1,0]$ defined by $f_i(u)$ and $f_j(u-[1,0])$, respectively, is valid on $A_{0,Y} \cup (A_{0,Y} + [1,0])$. Then, $\nu(X,Y) = \mathbf{1}' \cdot M_Y^{X-1} \cdot \mathbf{1} = \mathbf{1}' \cdot M_X^{Y-1} \cdot \mathbf{1}$ where $\mathbf{1}$ is the all-ones column vector of the appropriate dimension and prime denotes transpose. The matrix M_Y meets the conditions of the Perron-Frobenius theorem [1, p. 17], since it has nonnegative elements and is irreducible. The largest magnitude eigenvalue Λ_Y of M_Y is positive, real, and has multiplicity one, and $\lim_{X\to\infty} (\nu(X,Y))^{1/X} = \Lambda_Y$. Furthermore, any valid labeling of $A_{X,k(Y+1)-1}$ defines a valid labeling of $A_{X,Y} + i(Y+1)[0,1]$ whenever $0 \le i < k$, and therefore $\nu(X,kY) \le (\nu(X,Y))^k$. Hence, for any $Y \ge 0$,

$$\begin{aligned} C_T &= \lim_{X,k\to\infty} \frac{\log_2 \nu(X,kY)}{2k(X+1)(Y+1)} \\ &\le \frac{1}{2(Y+1)} \lim_{X\to\infty} \log_2 (\nu(X,Y))^{\frac{1}{X+1}} = \frac{\log_2 \Lambda_Y}{2(Y+1)}. \end{aligned}$$

Thus the sequence $\{\log_2 \Lambda_i/(2(i+1))\}$ converges to C_T from above. Using Λ_{13} above gives $C_T \le 0.634775895$.

IV. Hard-triangle constrained encoding

To obtain a lower bound on the capacity C_T, we introduce the notion of a hard-triangle constrained encoder. The encoder maps a random sequence $\hat{w}$ of independent bits with the probability of a 0 equal $1/2$ into a hard-triangle constrained labeling of T, Then we calculate the coding rate of the encoder. The coding rate is a quantity that measures the efficiency of the encoder, and is known [2, p. 27] to be upper-bounded by the capacity C_T.

A hard-triangle constrained *encoder* is an injection $\mathcal{E} : \{0,1\}^\infty \longrightarrow \bigcup_{S\subset T} L(S)$ and its inverse is called a *decoder*. The encoder $\mathcal{E}$ maps an infinite binary input sequence into a labeling of a subset of T. An encoder and decoder are together called a *coding algorithm*.

Theorem 1 *The hard-triangle constrained encoder $\mathcal{E}$ achieves a coding rate of $r = 0.628831217$, which is within 1% of the capacity.*

Corollary 2 *The capacity C_T of the hard-triangle constraint is bounded as $0.628831217 \le C_T \le 0.634775895$.*

References

[1] R. B. Bapat and T. E. S. Raghavan *Nonnegative Matrices and Applications.* Cambridge University Press, Cambridge, United Kingdom, 1997.

[2] C. E. Shannon A Mathematical Theory of Communication *Bell Syst. Tech. J.*, 27(3,4):379-423 and 623-656, July and October, 1948.