# Asymptotic Capacity of the Two-Dimensional Square Constraint*

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*Abstract* — **Two-dimensional run length limited codes satisfying the square constraint are considered. Let $S$ denote a square of area $A(S)$ and let $\alpha_n$ be a positive sequence satisfying $\lim_{n\to\infty}\alpha_n = \infty$. It is shown that the capacity $C_n$ corresponding to the set $\mathcal{S}_n = \alpha_n S \cap \mathbf{Z}^2$ asymptotically satisfies**

$$\lim_{n\to\infty} C_n \cdot \frac{\alpha_n^2}{\log_2 \alpha_n^2} = \frac{4}{A(S)} \ .$$

## I. Introduction

One-dimensional run length constraints are important in magnetic recording applications and two-dimensional run length constraints have recently gained interest due to optical recording applications [1, 2, 3]. A two-dimensional run length constraint requires that a binary labeling of the integer lattice $\mathbf{Z}^2$ have a specified minimum and maximum number of zeros between consecutive ones both horizontally and vertically. Additional constraints, such as run length constraints along diagonals can also be imposed in order to more accurately model optical recording devices. In this paper we examine the asymptotic behavior of the "square" constraint. The square constraint imposes the condition that for every "one" stored in the plane, it must be surrounded by a square of zeros of some given side length. As the side length of the square grows to infinity the amount of information that can be stored per unit area shrinks to zero. In other words the capacity of the constraint falls to zero. In this paper we determined the exact rate that the capacity of the square constraint falls to zero as a function of the area of the constraint.

## II. Definitions and Results

Let $\mathbf{R}^2$ denote the two-dimensional plane, and $\mathbf{Z}^2$ the two-dimensional integer lattice (i.e. $\mathbf{Z}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbf{Z}\}$).

Suppose that $\mathcal{V} \subset \mathbf{Z}^2$, such that $(0,0) \in \mathcal{V}$. The code $f : \mathbf{Z}^2 \to \{0,1\}$ *satisfies the constraint* $\mathcal{V}$ (or, $f$ defines a *valid labeling* of $\mathbf{Z}^2$ with respect to $\mathcal{V}$), if for every $\mathbf{x} \in \mathbf{Z}^2$

$$f(\mathbf{x}) = 1 \Rightarrow f(\mathbf{y}) = 0 \qquad \text{for } \forall \mathbf{y} \in \mathcal{V}+\mathbf{x},\ \ \mathbf{y} \neq \mathbf{x} \ . \quad (1)$$

A subset of $\mathbf{Z}^2$ of the form $\mathcal{R}_{(a,b)}^{(c,d)} = \{(\mathbf{x},\mathbf{y}) \in \mathbf{Z}^2 : a \le x \le c, b \le y \le d\}$ for some integers $a, b, c, d$, will be called a rectangle. A binary labeling of the rectangle $\mathcal{R}_{(a,b)}^{(c,d)}$is *valid* with respect to a given constraint $\mathcal{V}$, if the labeling can be extended to a labeling of $\mathbf{Z}^2$ satisfying the constraint $\mathcal{V}$. Let $N_{\mathcal{V}}(m,n)$ denote the number of valid labelings of the rectangle $\mathcal{R}_{(0,0)}^{(n,m)}$ with respect to $\mathcal{V}$. The capacity $C_{\mathcal{V}}$ corresponding to a set $\mathcal{V} \subset \mathbf{Z}^2$ including the origin is defined as

$$C_{\mathcal{V}} = \lim_{m,n\to\infty} \frac{\log_2 N_{\mathcal{V}}(m-1, n-1)}{mn} \ .$$

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The proof in [1] can be generalized to show that the above limit exists.

## III. The asymptotic capacity of the square constraint

In this section $S \subset \mathbf{R}^2$ will denote a square centered at the origin, whose sides are parallel to the coordinate axes. Let $\mathcal{S} = S \cap \mathbf{Z}^2$, and let $\alpha_n$ be a sequence of positive real numbers, such that $\lim_{n\to\infty}\alpha_n = \infty$. Consider the sequence of capacities $C_n$ corresponding to the constraints $\mathcal{S}_n = \alpha_n S \cap \mathbf{Z}^2$, as $n \to \infty$. In the main theorem of this section we determine the asymptotic rate that $C_n$ goes to zero as $n \to \infty$.

**Lemma 1** *Let $C_n$ denote the capacity corresponding to the constraint $\mathcal{S}_n$. Write $\mathcal{S}_n = \mathcal{R}_{(-d,-d)}^{(d,d)}$ for some integer $d$, and consider the set $\hat{\mathcal{S}}_n = \mathcal{R}_{(0,0)}^{(d,d)}$. For every positive integer $n$, $C_n$ satisfies the inequality*

$$C_n \le \frac{\log_2(A(\hat{\mathcal{S}}_n) + 1)}{A(\hat{\mathcal{S}}_n)} \ ,$$

*where $A(\hat{\mathcal{S}}_n)$ denotes the number of lattice points in $\hat{\mathcal{S}}_n$.*

**Lemma 2** *Let $C_n$ denote the capacity corresponding to the constraint $\mathcal{S}_n$. Write $\mathcal{S}_n = \mathcal{R}_{(-d,-d)}^{(d,d)}$ for some integer $d$, and consider the set $\hat{\mathcal{S}}_n = \mathcal{R}_{(0,0)}^{(d,d)}$. For $\forall \epsilon > 0, \forall \gamma \in \mathbf{Z}^+$ there exists $N$, such that for $\forall n > N$,*

$$C_n \ge \left(\frac{\gamma}{\gamma+1}\right)^2 \frac{\log_2 A(\hat{\mathcal{S}}_n))}{A(\hat{\mathcal{S}}_n)} - \epsilon. \qquad (2)$$

**Theorem 1** *Let $C_n$ denote the capacity corresponding to the constraint $\mathcal{S}_n = \alpha_n S \cap \mathbf{Z}^2$. Then,*

$$\lim_{n\to\infty} C_n \cdot \frac{\alpha_n^2}{\log_2 \alpha_n^2} = \frac{4}{A(S)} \ .$$

## References

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