# Randomly Chosen Index Assignments Are Asymptotically Bad for Uniform Sources

András Méhes and Kenneth Zeger[1]

Department of Electrical and Computer Engineering, University of California, San Diego, CA 92093-0407.

{andras,zeger}@code.ucsd.edu

*Abstract* — **It is known that among all redundancy free codes (or index assignments) the Natural Binary Code minimizes the mean squared error of the uniform source and uniform quantizer on a binary symmetric channel. We derive a code which maximizes the mean squared error, and demonstrate that the code is linear and its distortion is asymptotically equivalent, as the blocklength grows, to the expected distortion of an index assignment chosen uniformly at random.**

## I. Introduction

An index assignment is a mapping of source code symbols to channel code symbols. The usual goal of index assignment design for noisy channel vector quantizers is to minimize the end-to-end mean squared error (MSE) over all possible index assignments. The MSE is computed with respect to the statistics of both the source and the channel. Previous work has examined the theoretical and practical aspects of index assignment in noisy channel vector quantizer systems. In particular, it is known that the performance of such a system can be significantly affected by the choice of index assignment.

In this paper we present an index assignment (the Worst Code) which *maximizes* the MSE for uniform scalar quantization of a uniform source, and compare the performances of the best, worst, and randomly chosen index assignments. The MSE of the Worst Code asymptotically equals the expected performance of an index assignment chosen uniformly at random. This indicates that the majority of index assignments are asymptotically bad.

Among affine index assignments the Worst Code also maximizes the MSE of arbitrary binary lattice vector quantizers. For certain binary lattice vector quantizers, however, some non-affine index assignments achieve higher distortion than the Worst Code.

## II. Index Assignments

An index assignment is a permutation of $\mathbb{Z}_2^n$. An *affine index assignment* $\pi\colon \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ has the form $\pi(\mathbf{i}) = \mathbf{i}\mathsf{G} + \mathbf{t}$, where $\mathsf{G}$ is a binary nonsingular $n \times n$ *generator matrix*, $\mathbf{t}$ is an $n$-dimensional binary *translation vector*, and the arithmetic is performed in $\mathbb{Z}_2^n$. If $\mathbf{t} = \mathbf{0}$, then $\pi$ is said to be *linear*.

The *Natural Binary Code* $\pi_N$ is the identity index assignment $\pi_N(\mathbf{i}) = \mathbf{i}$. It is linear with generator matrix $\mathsf{G}_N = \mathsf{I}$, the identity matrix. We define the *Worst Code* $\pi_W$ as the linear index assignment with generator matrix

$$\mathsf{G}_W = \begin{bmatrix} 1 + \mathbf{1}\cdot\mathbf{1}' & \mathbf{1} \\ \mathbf{1}' & \mathsf{I} \end{bmatrix} \qquad \mathsf{G}_W^{-1} = \begin{bmatrix} 1 & \mathbf{1} \\ \mathbf{1}' & \mathsf{I} + \mathbf{1}'\cdot\mathbf{1} \end{bmatrix},$$

where $\mathbf{1}'$ denotes the transpose of a vector of all ones.

[1] The research was supported in part by the National Science Foundation.

## III. Summary of Results

**Theorem 1** *Suppose an integer $i$ is chosen uniformly at random from the set $\mathcal{S} = \{0, \dots, 2^n - 1\}$ and the $n$-bit word $\pi(\mathbf{i})$ is transmitted over a binary symmetric channel with bit error probability $\epsilon \in [0, 1/2]$, using an index assignment $\pi$. Then the resulting mean squared error $D$ satisfies*

$$\epsilon \frac{4^n - 1}{3} \le D \le \epsilon (1 - 2\epsilon)^{n-1} 4^{n-1} + \left(1 - (1 - 2\epsilon)^{n-1}\right) \frac{4^n - 1}{6},$$

*where the lower bound is achieved by the Natural Binary Code [1, 2] and the upper bound by the Worst Code.*

Let $D_{\min}$ and $D_{\max}$ respectively denote the distortions of the Natural Binary Code and the Worst Code (the lower and upper bounds in Theorem 1). The distortion of an index assignment chosen uniformly at random is obtained by averaging over all index assignments and equals

$$D_{\text{ave}} = \left(1 - (1 - \epsilon)^n\right) \frac{4^n + 2^n}{6}. \tag{1}$$

The values of $D_{\min}$ and $D_{\text{ave}}$ were reported in [3].

**Corollary 1** *For any fixed $n$, as $\epsilon \to 0$ the relative mean squared errors of the worst, average, and best index assignments for the uniform source obey the following ratios:*

$$D_{\max} : D_{\text{ave}} : D_{\min} \approx 1 : 1/2 : 1/n,$$

*and for any fixed $\epsilon$, as $n \to \infty$ the relative mean squared errors obey the ratios:*

$$D_{\max} : D_{\text{ave}} : D_{\min} = 1 : 1 : 2\epsilon.$$

Thus for asymptotically large block lengths on a fixed channel, the performance gain of the best index assignment over the worst and average ones is $1/2\epsilon$ which can be very significant. In this sense, a large fraction of index assignments can be considered "bad". Also, for any $\epsilon > 0$, the expected distortion of a randomly chosen index assignment asymptotically equals (as the blocklength grows) that of a worst index assignment.

## References

[1] T. R. Crimmins, H. M. Horwitz, C. J. Palermo, and R. V. Palermo, "Minimization of Mean-Square Error for Data Transmitted Via Group Codes," *IEEE Trans. Info. Theory*, vol. IT-15, pp. 72–78, January 1969.

[2] S. W. McLaughlin, D. L. Neuhoff, and J. J. Ashley, "Optimal Binary Index Assignments for a Class of Equiprobable Scalar and Vector Quantizers," *IEEE Trans. Info. Theory*, vol. IT-41, pp. 2031–2037, November 1995.

[3] Y. Yamaguchi and T. S. Huang, "Optimum Binary Fixed-Length Block Codes," Quarterly Progress Report 78, M.I.T. Research Lab. of Electronics, Cambridge, Mass., July 1965.