Affine Index Assignments for Binary Lattice Quantization with Channel Noise[1]

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Abstract — **A general formula is given for the MSE performance of affine index assignments for a binary symmetric channel with an arbitrary source and a binary lattice quantizer. The result is then used to compare some well-known redundancy free codes. The binary asymmetric channel is considered for a uniform input distribution and a class of affine codes.**

Two major issues in noisy channel vector quantization are complexity and sensitivity to channel errors. Structured vector quantizers and index assingments provide a low complexity solution for enhancing channel robustness.

A d-dimensional, n-bit *noisy channel VQ* with *index set* $\mathcal{I} = \{0, 1, \ldots, 2^n - 1\}$, and *code book* $\mathcal{C} = \{\mathbf{y}_i \in \mathbb{R}^d : i \in \mathcal{I}\}$ is a functional composition $Q = \mathcal{D} \circ \pi^{-1} \circ \eta \circ \pi \circ \mathcal{E}$, where $\mathcal{E}: \mathbb{R}^d \to \mathcal{I}$ is the *quantizer encoder*, $\mathcal{D}: \mathcal{I} \to \mathcal{C}$ is the *quantizer decoder*, $\pi: \mathcal{I} \to \mathcal{I}$ is the *index permutation*, and $\eta: \mathcal{I} \to \mathcal{I}$ is a random permutation representing the channel.

A *binary lattice quantizer* is a vector quantizer, whose codevectors are of the form $\mathbf{y}_i = \mathbf{y}_0 + \sum_{l=0}^{n-1} \mathbf{v}_l i_l$ for $i \in \mathcal{I}$, where the ordered set of vectors $\mathcal{V} = \{\mathbf{v}_l\}_{l=0}^{n-1}$ is called the *generating set*, and $i_l \in \{0, 1\}$ is the l^{th} bit in the binary expansion of the index i (here i_0 is the LSB). A binary lattice quantizer is equivalent to a direct sum quantizer (or multistage or residual quantizer) with two code vectors per stage. Examples include truncated lattice vector quantizers (e.g. uniform quantizers). A binary lattice VQ is similar to the non-redundant version of the LMBC-VQ (VQ by Linear Mappings of Block Codes) presented in [3].

An *affine index assignment* is an assignment of the form

$$\pi(i) = \vec{i}\mathrm{G} \oplus \vec{d}, \qquad \pi^{-1}(i) = (\vec{i} \oplus \vec{d})\mathrm{F}, \ \ (\mathrm{F} = \mathrm{G}^{-1})$$

where G is the *generator matrix*, $\vec{d}$ is the *translation vector*, and the operations are performed over $GF(2)$. Many popular redundancy free codes are affine, including the Natural Binary Code (NBC), the Folded Binary Code (FBC), and the Gray Code (GC).

For a given source X, the *Hadamard transform* of its distribution is defined as $\hat{P}_l = \sum_{i \in \mathcal{I}} \mathrm{P}[\mathcal{E}(\mathbf{X}) = i](-1)^{\langle \vec{i}, \vec{l} \rangle}$.

The MSE of a quantizer that satisfies the centroid condition, can be decomposed as $D = D_S + D_C$, where

$$\begin{aligned} D_S &= \sum_{i \in \mathcal{I}} \mathrm{E}\left[\|\mathbf{X} - \mathbf{y}_i\|^2 \mid \mathcal{E}(\mathbf{X}) = i\right] \mathrm{P}[\mathcal{E}(\mathbf{X}) = i] \\ D_C &= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \|\mathbf{y}_i - \mathbf{y}_j\|^2 \mathrm{P}[\mathcal{E}(\mathbf{X}) = i] \mathrm{P}[\pi(j) | \pi(i)]. \end{aligned}$$

Theorem 1 *The channel distortion of a 2^n point binary lattice vector quantizer with generating set $\{\mathbf{v}_l\}_{l=0}^{n-1}$, which uses an affine index assignment with generator matrix* G *to transmit across a binary symmetric channel with crossover probability q, is given by*

$$D_C = \frac{1}{4} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \langle \mathbf{v}_k, \mathbf{v}_l \rangle \hat{P}_{2^k + 2^l} \times \left(1 - 2(1-2q)^{w(f_{n-k}^T)} + (1-2q)^{w(f_{n-k}^T \oplus f_{n-l}^T)}\right),$$

where $w(\cdot)$ denotes Hamming weight, $f^T = [f_{1,k}, \ldots, f_{n,k}]$ is the k^{th} column of $\mathrm{F} = \mathrm{G}^{-1}$, $\hat{P}_l$ is the l^{th} component of the Hadamard transform of the induced discrete distribution on the encoder cells, and $\oplus$ indicates modulo 2 addition.

Let FBC* denote the "best" Folded Binary Code obtained by reordering the generating set $\mathcal{V}$ to minimize D_C, and let U be a uniform discrete random variable on the code points.

Corollary 1 *Given the conditions of Theorem 1 (and $q < 1/2$), $D_C^{(FBC^*)} > D_C^{(NBC)}$ if and only if*

$$\mathrm{Var}[Q(\mathbf{X})] + \mathrm{E}^2[Q(\mathbf{X}) - \mathrm{U}] > \frac{\max_{\mathbf{v} \in \mathcal{V}} \|\mathbf{v}\|^2}{\sum_{\mathbf{v} \in \mathcal{V}} \|\mathbf{v}\|^2} \mathrm{Var}[\mathrm{U}]$$

For a uniform (discrete) distribution on the code vectors, and a binary symmetric channel the NBC is the optimal index assignment [1], [2]. An *affine translate of the NBC* is an index assignment of the form $\pi(i) = \vec{i} \oplus \vec{d} = \pi^{-1}(i)$.

Theorem 2 *If a 2^n point binary lattice vector quantizer induces equiprobable encoder cells for a given source, and transmits an affine translation of the Natural Binary Code across a binary asymmetric channel with crossover probabilities $\mathrm{P}[1|0] = p$ and $\mathrm{P}[0|1] = q$, then the channel distortion is minimized if and only if the translation vector $\vec{d}$ satisfies*

$$\vec{d} = \operatorname*{argmin}_{i \in \mathcal{I}} \|\mathbf{y}_i - \mathrm{E}[\mathrm{U}]\|$$

References

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