Redundancy Free Codes for Binary Discrete Memoryless Channels *

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Abstract

Formulas are derived for the MSE performance of several classical redundancy free codes for a binary discrete memoryless channel with a uniform scalar source and a uniform quantizer. In particular these results generalize formulas previously obtained for the binary symmetric channel. It is also proven that although the Natural Binary Code is optimal for the binary symmetric channel, it is never optimal for any other binary channel. It is further proven that the Gray Code is never optimal for any binary channel. Finally, we introduce a new code, called the Odd-Even Code, which outperforms the NBC on every nonsymmetric binary channel and performs equally well on the BSC.

1. Introduction

Consider the problem of quantizing a real-valued random variable X by using a scalar quantizer with 2^n levels, where n bits are transmitted across a binary Discrete Memoryless Channel (DMC). A *redundancy free code* is a permutation of the 2^n possible n-bit binary words to be transmitted (i.e. a rate 1 channel code). This can be readily generalized to vector quantization. The model of the system we consider is shown in Figure 1.

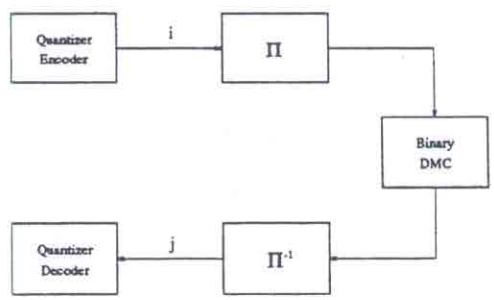

Figure 1: Communication System Block Diagram. π is the redundancy free code, also known as an index assignment.

What is the best redundancy free code to use? In general, this is an open problem in combined source/channel coding theory. An exhaustive search of all $(2^n)!$ different permutations is computationally infeasible for even relatively small quantizer codebooks. For example if $n \geq 4$ then there are at least 16! permutations, an extremely large number. Traditionally, there have been three important commonly studied codes for transmitting digitized analog data. These are the Natural Binary Code (NBC), Folded Binary Code (FBC), and Gray Code (GC) [8]. Strictly speaking, each of these is a class of codes parameterized by the number of binary digits transmitted.

Some heuristic techniques have been developed for finding good suboptimal codes [1], [2], [3]. However, very few theoretical results exist for finding the mean square distortion for particular redundancy free codes or for finding optimal codes. For very large quantizer codebooks, existing iterative design procedures are too complex.

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Huang [4, 5, 6] gave a formula for the MSE in using the NBC on a Binary Symmetric Channel (BSC) with a uniform source and a uniform quantizer. McLaughlin, Ashley, and Neuhoff [9] recently used a vector space approach to prove that the NBC achieves the lowest possible MSE for a BSC.

A formula is also known for the MSE of the GC on a BSC. Other than the BSC there are not known any such formulas for any other binary discrete memoryless channel, nor for the FBC on a BSC. The previously known BSC formulas are given Propositions 1 and 2.

Our main result in this paper is to derive formulas for the NBC, GC, and FBC for an arbitrary binary DMC, assuming a uniform source and uniform scalar quantizer. Of course, in particular our results hold for such channels as the BSC and the Z-channel. In addition, we introduce a new code, which we call the *Odd-Even Code (OEC)*, and derive a formula for its MSE on a binary DMC. We show that the OEC is equivalent in performance to the NBC on a BSC, but is always superior to the NBC for any binary channel other than the BSC. For this reason it would be advantageous in general to use the OEC rather than the NBC. We also observed experimentally that the OEC is the best code overall for most useful binary channels for the special cases of 1,2, and 3 bit codes. Finally, we prove that the GC is *never the best code* for any binary channel, perhaps a somewhat surprising result.

Theorems 1-4 give our MSE formulas and Theorem 5 provides a comparison of the codes' performances. Due to space constraints, we only include a proof of Theorem 1 in this paper. The remaining proofs will be given in a future publication.

2. Definitions

A *redundancy free code* with 2^n levels is a permutation Π_n of the set $\{0, 1, \ldots, 2^n - 1\}$.

- The *Natural Binary Code* is the identity mapping:

$$\Pi_n^{(NBC)}(i) = i \qquad 0 \le i \le 2^n - 1.$$

- The *Folded Binary Code* (or *Sign-Magnitude Code*) is defined as

$$\Pi_n^{(FBC)}(i) = \begin{cases} 2^{n-1} - 1 - i & 0 \le i \le 2^{n-1} - 1 \\ i & 2^{n-1} \le i \le 2^n - 1 \end{cases}.$$

i	$\Pi_4^{(NBC)}(i)$		$\Pi_4^{(FBC)}(i)$		$\Pi_4^{(GC)}(i)$		$\Pi_4^{(OEC)}(i)$	
0	0	0000	7	0111	0	0000	1	0001
1	1	0001	6	0110	1	0001	3	0011
2	2	0010	5	0101	3	0011	5	0101
3	3	0011	4	0100	2	0010	7	0111
4	4	0100	3	0011	6	0110	9	1001
5	5	0101	2	0010	7	0111	11	1011
6	6	0110	1	0001	5	0101	13	1101
7	7	0111	0	0000	4	0100	15	1111
8	8	1000	8	1000	12	1100	0	0000
9	9	1001	9	1001	13	1101	2	0010
10	10	1010	10	1010	15	1111	4	0100
11	11	1011	11	1011	14	1110	6	0110
12	12	1100	12	1100	10	1010	8	1000
13	13	1101	13	1101	11	1011	10	1010
14	14	1110	14	1110	9	1001	12	1100
15	15	1111	15	1111	8	1000	14	1110

Table 1: Examples of Common Redundancy Free Codes

- The *Gray Code* (or *Reflected Binary Code*) is defined recursively as

$$\Pi_n^{(GC)}(i) = \begin{cases} \Pi_{n-1}^{(GC)}(i) & i \le 2^{n-1} - 1 \\ 2^{n-1} + \Pi_{n-1}^{(GC)}(2^n - 1 - i) & 2^{n-1} \le i \end{cases},$$

with the initial condition $\Pi_0^{(GC)}(0) = 0$.

- The *Odd-Even Code* is defined as

$$\Pi_n^{(OEC)}(i) = \begin{cases} 2i + 1 & 0 \le i \le 2^{n-1} - 1 \\ 2i - 2^n & 2^{n-1} \le i \le 2^n - 1 \end{cases}.$$

Note that the OEC can be obtained from the NBC by the relation

$$\Pi_n^{(OEC)}(i) = \text{rotate}[\Pi_n^{(NBC)}(i)] \oplus 1,$$

where rotate[j] is a left cyclic shift of the binary representation of j and $\oplus$ is componentwise modulo-2 addition.

- The *One's Complement* code $\overline{X}$, of a code X, is defined by

$$\Pi_n^{\overline{X}}(i) = 2^n - 1 - \Pi_n^{(X)}(i) \qquad 0 \le i \le 2^n - 1$$

A code can be conveniently viewed by a listing of the binary representations of the numbers $\Pi_n(0), \Pi_n(1), \ldots, \Pi_n(2^n - 1)$. An example for $n = 4$ is given in Table 1.

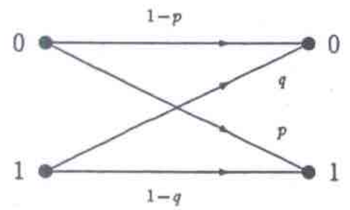


Figure 2: BAC with $\mathrm{P}(1|0) = p$ and $\mathrm{P}(0|1) = q$

3. Mean Square Error Formulas

Let i and j be binary words and let $\mathrm{P}(j|i) = \mathrm{P}(j \text{ received} \mid i \text{ sent})$ be the word transition probabilities of a binary discrete memoryless channel. The MSE of a 2^n level scalar quantizer with encoder cells $R_0, \ldots, R_{2^n-1}$ that satisfies the centroid condition, can be decomposed into a source component and a channel component as $D = D_S + D_C$, where

$$\begin{aligned} D_S &= \sum_{i=0}^{2^n-1} \mathrm{E}[(X-y_i)^2 \mid X \in R_i]\mathrm{P}(X \in R_i) \\ D_C &= \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} (y_i - y_j)^2 \mathrm{P}(X \in R_i)\mathrm{P}(j|i) \end{aligned}$$

If the source is uniform on the interval $[a, b]$ and a uniform quantizer is used then

$$\begin{aligned} D_S &= \frac{\Delta^2}{12} \\ \mathrm{P}(X \in R_i) &= 2^{-n} \end{aligned}$$

where $\Delta = (b-a)/2^n$. Formulas for D_C are derived here for various important codes and for all memoryless binary channels, such as shown in Figure 2. The most common examples are the BSC (with $p = q$), and the Z-channel (with $p = 0$ or $q = 0$).

If maximum likelihood decoding is used and a 0 (resp. 1) is received, then a 0 (resp. 1) is decoded as the estimate of the transmitted binary symbol provided $P(0|0) > P(0|1)$ (resp. $P(1|1) > P(1|0)$), i.e. $p + q < 1$. If $p + q > 1$ then the roles of 0 and 1 are reversed and the channel performs the same as a binary channel with $P(0|1) = 1 - q$ and $P(1|0) = 1 - p$. Hence, throughout this paper we assume $p + q < 1$.

Let us denote the $2^n \times 2^n$ channel transition probability matrix by $T_n = [t_{i,j}^{(n)}]$, where

$$t_{i,j}^{(n)} = \mathrm{P}(j|i); \qquad i, j \in \{0, \ldots, 2^n - 1\} \equiv \{0,1\}^n.$$

The memoryless property of the channel yields the following recursion

$$T_{n+1} = \begin{bmatrix} (1-p)T_n & pT_n \\ qT_n & (1-q)T_n \end{bmatrix}, \tag{1}$$

(with $T_0 = [1]$) since $t_{xi,yj}^{(n+1)} = \mathrm{P}(y|x) \cdot t_{i,j}^{(n)}$ for $x, y \in \{0,1\}$, where xi (and similarly yj) denotes concatenation. It will be convenient to define another $2^n \times 2^n$ matrix $S_n = [s_{i,j}^{(n)}]$, where

$$s_{i,j}^{(n)} = \left(\Pi_n^{-1}(i) - \Pi_n^{-1}(j)\right)^2; \qquad i, j \in \{0, \ldots, 2^n - 1\}. \tag{2}$$

Then we can write

$$\begin{aligned} D_C &= 2^{-n}\Delta^2 \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} (i-j)^2 \mathrm{P}(\Pi_n(j)|\Pi_n(i)) \quad (3) \\ &= 2^{-n}\Delta^2 \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} s_{i,j}^{(n)} t_{i,j}^{(n)}. \quad (4) \end{aligned}$$

Definition 1 *Let $\mathcal{C}_n = \{A_n = [a_{i,j}^{(n)}] \mid a_{i,j}^{(n)} \in \mathcal{R};\ i,j \in \{0, \ldots, 2^n - 1\}\}$ be the class of $2^n \times 2^n$ real matrices. The linear operator $\mathrm{L}(\cdot) : \mathcal{C}_n \mapsto \mathcal{R}$ is given by*

$$\mathrm{L}(A_n) \triangleq \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} a_{i,j}^{(n)} t_{i,j}^{(n)}, \tag{5}$$

where $A_n \in \mathcal{C}_n$ and $[t_{i,j}^{(n)}]$ is the channel transition matrix.

An important property of this operator is due to the recursive structure of T_n:

Lemma 1 *For every matrix*

$$A_{n+1} = \begin{bmatrix} A_n^{(00)} & A_n^{(01)} \\ A_n^{(10)} & A_n^{(11)} \end{bmatrix} \in \mathcal{C}_{n+1},$$

where $A_n^{(00)}, A_n^{(01)}, A_n^{(10)}, A_n^{(11)} \in \mathcal{C}_n$, we have

$$\begin{aligned} \mathrm{L}(A_{n+1}) &= (1-p)\mathrm{L}(A_n^{(00)}) + p\mathrm{L}(A_n^{(01)}) \\ &+ q\mathrm{L}(A_n^{(10)}) + (1-q)\mathrm{L}(A_n^{(11)}). \quad (6) \end{aligned}$$

Since $S_n \in \mathcal{C}_n$ the operator notation can be used to write

$$D_C = 2^{-n}\Delta^2 \mathrm{L}(S_n).$$

Proposition 1 (Huang[4][5]) *The channel distortion of a uniform 2^n level scalar quantizer for a uniform source on the interval $[a, b]$, which transmits the Natural Binary Code (NBC) across a Binary Symmetric Channel (BSC) with $\mathrm{P}(1|0) = \mathrm{P}(0|1) = p$, is given by*

$$D_C^{(NBC)} = \frac{(b-a)^2}{3} p \left(1 - \frac{1}{4^n}\right).$$

The NBC has recently been proven optimal under the assumptions in Proposition 1 by McLaughlin, Ashley and Neuhoff [9].

Proposition 2 (Huang[5]) *The channel distortion of a uniform 2^n level scalar quantizer for a uniform source on the interval $[a,b]$, which transmits the Gray Code (GC) across a Binary Symmetric Channel (BSC) with* $\mathrm{P}(1|0) = \mathrm{P}(0|1) = p$, *is given by*

$$D_C^{(GC)} = \frac{(b-a)^2}{4^n}\left[\frac{4^n-1}{6} - \left(\frac{1-2p}{2}\right)\frac{4^n-(1-2p)^n}{4-(1-2p)}\right].$$

Theorem 1 *The channel distortion of a uniform 2^n level scalar quantizer for a uniform source on the interval $[a,b]$, which transmits the Natural Binary Code (NBC) across a binary discrete memoryless channel with* $\mathrm{P}(1|0) = p$ *and* $\mathrm{P}(0|1) = q$, *is given by*

$$D_C^{(NBC)} = \frac{(b-a)^2}{6}\left[\alpha\left(1-\frac{1}{4^n}\right) + \beta^2\left(1-\frac{3}{2^n}+\frac{2}{4^n}\right)\right]$$

where $\alpha = p+q$ and $\beta = q-p$.

Proof

First we write the NBC recursively as [1]:

$$\Pi_{n+1}^{-1}(i) = \begin{cases} \Pi_n^{-1}(i) & 0 \le i \le 2^n-1 \\ 2^n + \Pi_n^{-1}(i-2^n) & 2^n \le i \le 2^{n+1}-1 \end{cases}$$

We introduce a matrix $U_n = [u_{i,j}^{(n)}]$ defined by

$$u_{i,j}^{(n)} = \Pi_n^{-1}(i) - \Pi_n^{-1}(j); \qquad i,j \in \{0,\ldots,2^n-1\}. \quad (7)$$

Using (2) and (7) we obtain

- For $i,j \in \{0,\ldots,2^n-1\}$
$$s_{i,j}^{(n+1)} = \left(\Pi_n^{-1}(i) - \Pi_n^{-1}(j)\right)^2 = s_{i,j}^{(n)}$$
- For $i \in \{0,\ldots,2^n-1\}$; $j \in \{2^n,\ldots,2^{n+1}-1\}$
$$\begin{aligned} s_{i,j}^{(n+1)} &= \left(\Pi_n^{-1}(i) - 2^n - \Pi_n^{-1}(j-2^n)\right)^2 \\ &= \left(\Pi_n^{-1}(i) - \Pi_n^{-1}(j-2^n)\right)^2 \\ &- 2\cdot 2^n\left(\Pi_n^{-1}(i) - \Pi_n^{-1}(j-2^n)\right) + 4^n \\ &= s_{i,j-2^n}^{(n)} - 2^{n+1}u_{i,j-2^n}^{(n)} + 4^n \end{aligned}$$
- For $i \in \{2^n,\ldots,2^{n+1}-1\}$; $j \in \{0,\ldots,2^n-1\}$
$$\begin{aligned} s_{i,j}^{(n+1)} &= \left(2^n + \Pi_n^{-1}(i-2^n) - \Pi_n^{-1}(j)\right)^2 \\ &= \left(\Pi_n^{-1}(i-2^n) - \Pi_n^{-1}(j)\right)^2 \\ &+ 2\cdot 2^n\left(\Pi_n^{-1}(i-2^n) - \Pi_n^{-1}(j)\right) + 4^n \\ &= s_{i-2^n,j}^{(n)} + 2^{n+1}u_{i-2^n,j}^{(n)} + 4^n \end{aligned}$$
- For $i,j \in \{2^n,\ldots,2^{n+1}-1\}$
$$\begin{aligned} s_{i,j}^{(n+1)} &= \left(2^n + \Pi_n^{-1}(i-2^n) - 2^n - \Pi_n^{-1}(j-2^n)\right)^2 \\ &= s_{i-2^n,j-2^n}^{(n)} \end{aligned}$$

From these we can immediately deduce that

$$S_{n+1} = \begin{bmatrix} S_n & S_n - 2^{n+1}U_n + 4^n\mathbf{1}_n \\ S_n + 2^{n+1}U_n + 4^n\mathbf{1}_n & S_n \end{bmatrix} \quad (8)$$

where $\mathbf{1}_n$ is the $2^n \times 2^n$ matrix of all ones. In order to obtain a solvable recursion for S_{n+1} we must first explicitly solve for U_n. This can be accomplished recursively.

- For $i,j \in \{0,\ldots,2^n-1\}$
$$u_{i,j}^{(n+1)} = \Pi_n^{-1}(i) - \Pi_n^{-1}(j) = u_{i,j}^{(n)}$$
- For $i \in \{0,\ldots,2^n-1\}$; $j \in \{2^n,\ldots,2^{n+1}-1\}$
$$u_{i,j}^{(n+1)} = \Pi_n^{-1}(i) - 2^n - \Pi_n^{-1}(j-2^n) = u_{i,j-2^n}^{(n)} - 2^n$$
- For $i \in \{2^n,\ldots,2^{n+1}-1\}$; $j \in \{0,\ldots,2^n-1\}$
$$u_{i,j}^{(n+1)} = 2^n + \Pi_n^{-1}(i-2^n) - \Pi_n^{-1}(j) = u_{i-2^n,j}^{(n)} + 2^n$$
- For $i,j \in \{2^n,\ldots,2^{n+1}-1\}$
$$u_{i,j}^{(n+1)} = u_{i-2^n,j-2^n}^{(n)}$$

Thus we have

$$U_{n+1} = \begin{bmatrix} U_n & U_n - 2^n\mathbf{1}_n \\ U_n + 2^n\mathbf{1}_n & U_n \end{bmatrix}.$$

Using Lemma 1 we have

$$\begin{aligned} \mathrm{L}(U_{n+1}) &= 2\mathrm{L}(U_n) + (q-p)4^n & (9) \\ \mathrm{L}(S_{n+1}) &= 2\mathrm{L}(S_n) + 2^{n+1}(q-p)\mathrm{L}(U_n) \\ &\quad +(p+q)8^n & (10) \end{aligned}$$

The initial conditions $\mathrm{L}(S_0) = \mathrm{L}(U_0) = 0$ we obtained from the definitions of S_n and U_n. Applying these, solving (9), and then substituting and solving (10), one obtains

$$\mathrm{L}(U_n) = (q-p)\binom{2^n}{2} \quad (11)$$

and

$$\mathrm{L}(S_n) = (p+q)\binom{2^n+1}{3} + (p-q)^2\binom{2^n}{3}. \quad (12)$$

[1] For notational simplicity here and throughout the proofs, the symbol Π_n^{-1} denotes the inverse of the actual permutation in question.

□

It is interesting to note that as $n \to \infty$, $D_C^{(NBC)} \to \frac{(b-a)^2}{6}\left[\alpha+\beta^2\right]$.

Theorem 2 *The channel distortion of a uniform 2^n level scalar quantizer for a uniform source on the interval $[a,b]$, which transmits the Folded Binary Code (FBC) across a binary discrete memoryless channel with $P(1|0)=p$ and $P(0|1)=q$, is given by*

$$D_C^{(FBC)} = \frac{(b-a)^2}{6}\left[\alpha\left(1-\frac{1}{4^n}\right)+\alpha(1-\alpha)\left(\frac{1}{4}-\frac{1}{4^n}\right)+\beta^2\left(\frac{1}{4}-\frac{3}{2^{n+1}}+\frac{2}{4^n}\right)-\alpha\beta\left(\frac{3}{4}-\frac{3}{2^{n+1}}\right)\right].$$

where $\alpha = p+q$ and $\beta = q-p$.

It follows that as $n\to\infty$, we have $D_C^{(FBC)} \to \frac{(b-a)^2}{6}\left[\alpha+\frac{\alpha(1-\alpha)}{4}+\frac{\beta^2}{4}-\frac{3\alpha\beta}{4}\right]$.

Theorem 3 *The channel distortion of a uniform 2^n level scalar quantizer for a uniform source on the interval $[a,b]$, which transmits the Gray Code (GC) across a binary discrete memoryless channel with $P(1|0)=p$ and $P(0|1)=q$, is given by*

$$D_C^{(GC)} = \frac{(b-a)^2}{4^n}\left(\frac{4^n-1}{6}-\frac{\gamma}{2}\frac{4^n-\gamma^n}{4-\gamma}+\frac{\beta}{2(2-\beta)}\cdot\left[\frac{4^n-1}{3}-\gamma\frac{4^n-\gamma^n}{4-\gamma}-\frac{(2\beta)^n-1}{2\beta-1}+\gamma\frac{(2\beta)^n-\gamma^n}{2\beta-\gamma}\right]\right)$$

where $\beta = q-p$ and $\gamma = 1-p-q$.

In the limit one gets $D_C^{(GC)} \to (b-a)^2\frac{4\alpha}{3(3+\alpha)(2-\beta)}$ as $n\to\infty$.

Theorem 4 *The channel distortion of a uniform 2^n level scalar quantizer for a uniform source on the interval $[a,b]$, which transmits the Odd Even Code (OEC) across a binary discrete memoryless channel with $P(1|0)=p$ and $P(0|1)=q$, is given by*

$$D_C^{(OEC)} = \frac{(b-a)^2}{6}\left[\alpha\left(1-\frac{1}{4^n}\right)-\beta^2\left(\frac{1}{2}-\frac{2}{4^n}\right)\right].$$

where $\alpha = p+q$ and $\beta = q-p$.

Thus $D_C^{(OEC)} \to \frac{(b-a)^2}{6}\left[\alpha-\frac{\beta^2}{2}\right]$, as $n\to\infty$.

4. Comparison

We note, that since α, γ, and β^2 are symmetric in p and q, so are the formulas for the distortions of the NBC and the OEC. Hence the one's complements of these codes yield the same performances as the codes themselves. This does not, however, hold for the FBC and the GC, since they contain odd powers of β, and β is not symmetric in p and q. Let us therefore examine the one's complement of these codes, denoted by $\overline{FBC}$ and $\overline{GC}$. Formulas for $D_C^{(\overline{FBC})}$ and $D_C^{(\overline{GC})}$ can be obtained by exchanging p and q (or equivalently, changing the sign of β) in the corresponding equations for $D_C^{(FBC)}$ and $D_C^{(GC)}$, respectively.

Theorem 5 *Given a uniform 2^n level scalar quantizer for a uniform source on the interval $[a,b]$, the channel distortions of the Natural Binary Code (NBC), the Folded Binary Code (FBC), the Gray Code (GC) and the Odd-Even Code (OEC) on a binary discrete memoryless channel with $P(1|0)=p$ and $P(0|1)=q$, satisfy (assuming $0\neq q\geq p$ and $p+q<1$):*

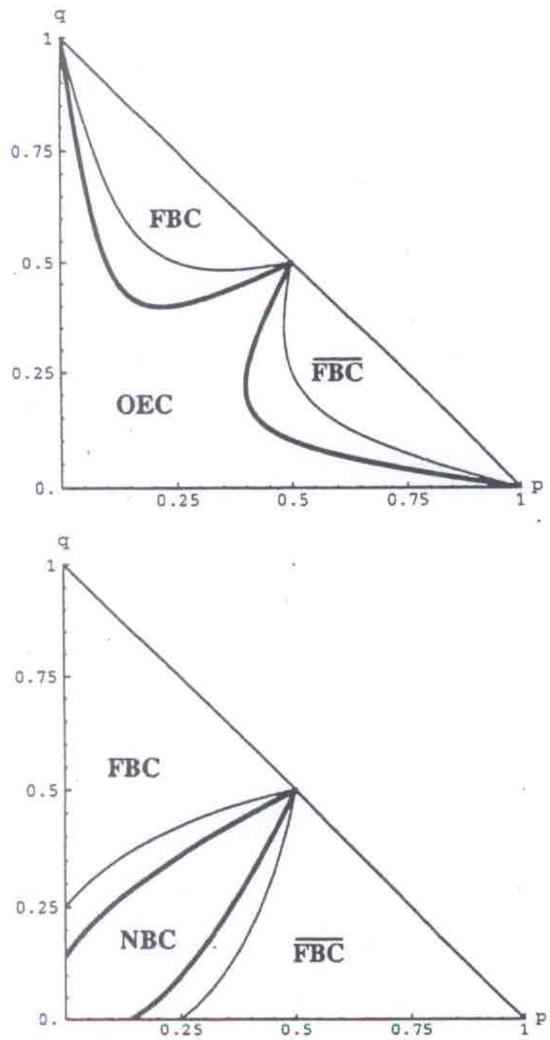
$$\begin{array}{llll}
(i)\ D_C^{(\overline{FBC})} & \geq & D_C^{(FBC)} \quad \forall n>1, \forall p,q & (13)\\
(ii)\ D_C^{(GC)} & \geq & D_C^{(\overline{GC})} \quad \forall n>1,\ \forall p,q & (14)\\
(iii)\ D_C^{(NBC)} & \geq & D_C^{(OEC)} \quad \forall n>1, \forall p,q & (15)\\
(iv)\ D_C^{(FBC)} & > & D_C^{(OEC)} & (16)\\
\forall n>1 & if & p+q+5p^2-8pq-q^2\geq 0 & \\
\forall n & \leq & 1+\log_2\frac{(p+q)(p+q-1)}{p+q+5p^2-8pq-q^2} & \\
 & if & p+q+5p^2-8pq-q^2<0 & \\
 & and & p+q+2p^2-5pq-q^2\geq 0, & \\
(v)\ D_C^{(FBC)} & > & D_C^{(NBC)} & (17)\\
\forall n>1 & if & p+q-p^2+4pq-7q^2\geq 0 & \\
\forall n & \leq & 1+\log_2\frac{(p+q)(p+q-1)}{p+q-p^2+4pq-7q^2} & \\
 & if & p+q-p^2+4pq-7q^2<0 & \\
 & and & p+q-p^2+pq-4q^2\geq 0, & \\
(vi)\ D_C^{(\overline{GC})} & > & D_C^{(FBC)} \quad \forall n>2, \forall p,q & (18)
\end{array}$$

The inequalities (i),(ii),(iii) hold with equality iff $p=q$.

The regions of the p vs. q plane identified in the theorem are shown in Figure 3.

Corollary 1 *The OEC has the same performance as the NBC on a BSC but is strictly better than the NBC for all* *other binary channels. Thus the NBC is not optimal on a binary channel unless the channel is symmetric.*

Corollary 2 *The GC is never optimal on a binary channel.*



(a)

(b)

Figure 3: Illustration for Theorem 5. In each graph two codes are compared. The region where one of them is uniformly (i.e. $\forall n$) better than the other is marked by the name of the superior code. (a) *FBC* vs. *OEC*. (b) *FBC* vs. *NBC*.

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