Asymptotic Entropy Constrained Performance of Tesselating and Universal Randomized Lattice Quantization

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ABSTRACT

Two results are given. First, using a result of Csiszár, the asymptotic (i.e., high resolution/low distortion) performance for entropy constrained tessellating vector quantization, heuristically derived by Gersho, is proven for all sources with finite differential entropy. This implies, using Gersho's Conjecture and Zador's formula, that tessellating vector quantizers are asymptotically optimal for this broad class of sources, and generalizes a rigorous result of Gish and Pierce from the scalar to vector case. Second, the asymptotic performance is established for Zamir and Feder's randomized lattice quantization. With the only assumption that the source has finite differential entropy, it is proven that the low distortion performance of the Zamir-Feder universal vector quantizer is asymptotically the same as that of the deterministic lattice quantizer.

SUMMARY

Let Q_N^k denote an N-level k-dimensional vector quantizer, and let X^k be the k-dimensional random vector with density f and differential entropy $h(f)$ to be quantized. Let the rth power quantization distortion be defined in the usual way,

$$D_r(Q_N^k(X^k)) = \frac{1}{k}E\|X^k - Q_N^k(X^k)\|^r,$$

where $\|\cdot\|$ denotes the Euclidian norm, and $r > 0$. Denote the Shannon entropy of Q_N^k by $H(Q_N^k)$, and for $H > 0$ let

$$D_e(H,k,r) = \inf_N \inf_{H(Q_N^k(X^k))\leq H} D_r(Q_N^k(X^k)), \tag{1}$$

the distortion of an optimal k-dimensional vector quantizer with entropy H. Gersho [2] heuristicly derived the asymptotic performance of quantizers given by the tessellation of $\mathcal{R}^k$ by a convex polytope P. He found that if $Q_{d,P}^k$ denotes the tesselating quantizer with rth power distortion d, then

$$\lim_{d\to 0} d\, 2^{\frac{r}{k}H(Q_{d,P}^k)} = l(P)\, 2^{\frac{r}{k}h(f)}, \tag{2}$$

where $l(P)$ is the normalized rth moment of P. We prove (2) in great generality. Our Theorem 1 establishes the asymptotic entropy constrained performance of lattice quantizers without any smoothness or compact support condition on the density. Thus the often quoted formula

$$\lim_{\Delta\to 0}[H(Q_\Delta) + \frac{1}{2}\log 12 D_2(Q_\Delta)] = h(f), \tag{3}$$

on the asymptotics of uniform quantizers is proved for all densities such that Q_Δ has finite Shannon entropy for some step size Δ, and $h(f) < \infty$, strengthening Gish and Pierce's result.

Ziv [4] introduced a randomized ("dithered") quantization scheme for scalar random variables which was extended by Zamir and Feder [3] for lattice vector quantizers.

We prove in Theorem 2 that for a large class of densities the asymptotic performance of the randomized lattice quantizer and the asymptotic performance of the ordinary lattice quantizer are the same.

Theorem 1 *If $|h(f)| < \infty$ and $H(Q_{d,P}^k(X^k)) < \infty$ for some $d > 0$, then*

$$\lim_{d\to 0} d\, 2^{\frac{r}{k}H(Q_{d,P}^k)} = l(P)\, 2^{\frac{r}{k}h(f)}. \tag{4}$$

Furthermore, if Zador's formula holds for f, $l(P) = C(k,r)$, and Gersho's conjecture holds, then

$$\lim_{d\to 0}\frac{D_e(H(Q_{d,P}^k),k,r)}{d} = 1, \tag{5}$$

i.e., the quantizer $Q_{d,P}^k$ is asymptotically optimal.

A standard technique using the vector Shannon lower bound on the kth order rate-distortion function $R_k(d)$ then gives for mean squared distortion

$$\limsup_{d\to 0}[\frac{1}{2}H(Q_{d,P}^k) - R_k(d)] \leq \frac{1}{2}\log 2\pi e l(P). \tag{6}$$

The condition for (6) to hold is that $E\|X^k\|^2 < \infty$, $|h(f)| < \infty$, and $H(Q_{d,P}^k(X^k)) < \infty$ for some $d > 0$.

Theorem 2 *Suppose the conditions of Theorem 1 hold. Then the rate $\bar{H}(Q_{d,V}^k)$ of the randomized lattice quantizer with basic cell V and rth power distortion d satisfies*

$$\lim_{d\to 0} d\, 2^{\frac{r}{k}\bar{H}(Q_{d,V}^k)} = l(V)\, 2^{\frac{r}{k}h(f)}. \tag{7}$$

i.e., the asymptotic performance of the randomized lattice quantizer is the same as the asymptotic performance of the ordinary (non-randomized) lattice quantizer given by (4).

Corollary 1 *For $r = 2$, $|h(f)| < \infty$, and $E\|X^k\|^2 < \infty$,*

$$\limsup_{d\to 0}\frac{1}{k}\bar{H}(Q_{d,V}^k) - R_k(d) \leq \frac{1}{2}\log 2\pi e l(V). \tag{8}$$

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