# On Source Coding with Side Information for General Distortion Measures


Tamás Linder
Dept. of Math. and Stat.
Queen's University
Kingston, Canada K7L 3N6
Email linder@mast.queensu.ca

Ram Zamir
Dept. of EE - Systems
Tel-Aviv University
Tel-Aviv, 69978, Israel
Email zamir@eng.tau.ac.il

Kenneth Zeger [1]
Dept. of ECE
University of California, San Diego
La Jolla CA 92093-0407 USA
Email zeger@ucsd.edu



*Abstract* — **High resolution bounds in lossy coding of a real memoryless source are derived when side information is present. We consider locally quadratic distortion measures which may be *functions of the side information* and give asymptotically tight bounds on the conditional rate-distortion function $R_{X|Y}(D)$ (the side information is available at both the encoder and the decoder) and on the Wyner-Ziv rate-distortion function $R^{\mathrm{WZ}}(D)$ (the side information is only available at the decoder). The rate loss $R^{\mathrm{WZ}}(D)-R_{X|Y}(D)$ is also investigated.**


## I. Introduction

Consider the source coding scenario depicted in Figure 1. The sequence $\{(X_k, Y_k)\}$ consists of independent and identically distributed copies of a pair of real random variables $(X, Y)$, where $X$ is called the *source* and $Y$ is called the *side information.* Fidelity of the reconstruction $\{\hat{X}_k\}$ is measured by means of a nonnegative single-letter distortion measure $d(x, y, \hat{x})$ which depends on the current value $y$ of the side information (context).

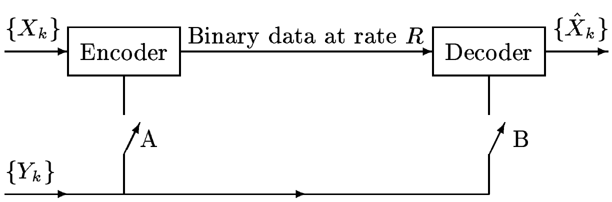


Figure 1: Source coding with side information

When switches A and B are closed, both the encoder and the decoder have access to the side information. Let $R_{X|Y}(D)$ denote the minimum achievable rate at distortion $D$. Then

$$R_{X|Y}(D) = \inf_{\hat{X}} I(X; \hat{X}|Y),$$

where the infimum of the conditional mutual information $I(X; \hat{X}|Y)$ is taken over all joint distributions of $(Y, X, \hat{X})$ such that $\mathbf{E}[d(X, Y, \hat{X})] \le D$.

If switch A is open and switch B is closed, only the decoder knows the side information. Let $R^{\mathrm{WZ}}(D)$ denote the minimum rate achievable at distortion $D$. Wyner and Ziv showed that

$$R^{\mathrm{WZ}}(D) = \inf_{Z} I(X; Z|Y),$$

where $Z$ is a real random variable, and where the infimum is taken over all joint distributions of $(X, Y, Z)$ such that $Y$ and $Z$ are conditionally independent given $X$, and there exists a measurable function $f(Y, Z)$ with $\mathbf{E}[d(X, Y, f(Y, Z))] \le D$.

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## II. Results

We will assume that $d(x, y, \hat{x})$ is a sufficiently smooth function such that $d(x, y, \hat{x}) = 0$ if and only if $x = \hat{x}$, and if $|x - \hat{x}|$ is small then the behavior of $d(x, y, \hat{x})$ is determined by the second order term in its Taylor expansion with respect to $\hat{x}$ around $(x, y, x)$. That is,

$$d(x, y, \hat{x}) = m(x, y)(x - \hat{x})^2 + o(|x - \hat{x}|^2)$$

as $|x - \hat{x}| \to 0$, where $m(x, y) = \frac{1}{2}(\partial^2/\partial\hat{x}^2) d(x, y, \hat{x})|_{\hat{x}=x}$. Assume $X$ has a density, finite variance and finite differential entropy $h(X)$, and suppose $I(X; Y) < \infty$.

**Theorem 1** *As $D \to 0$, the asymptotic behavior of $R_{X|Y}(D)$ is given by*

$$R_{X|Y}(D) = h(X|Y) - \frac{1}{2}\log(2\pi e D) + \frac{1}{2}\mathbf{E}\big[\log m(X, Y)\big] + o(1),$$

*where $o(1) \to 0$ as $D \to 0$.*

**Theorem 2** *As $D \to 0$, $R^{\mathrm{WZ}}(D)$ is upper bounded as*

$$R^{\mathrm{WZ}}(D) \le h(X|Y) - \frac{1}{2}\log(2\pi e D) + \frac{1}{2}\mathbf{E}\big[\log \overline{m}(X)\big] + o(1),$$

*where $\overline{m}(X) = \mathbf{E}\big[m(X, Y)|X\big]$. If additionally we assume that $X$ and $Y$ are independent, then the upper bound is asymptotically tight, i.e.,*

$$R^{\mathrm{WZ}}(D) = h(X) - \frac{1}{2}\log(2\pi e D) + \frac{1}{2}\mathbf{E}\big[\log \overline{m}(X)\big] + o(1).$$

We conjecture that the asymptotic upper bound in Theorem 2 is tight even for dependent $X$ and $Y$.

The nonnegative rate loss $R^{\mathrm{WZ}}(D) - R_{X|Y}(D)$ was investigated in [1] and it was established there that for difference distortion measures which do not depend on the side information (i.e., when $d(x, \hat{x}) = \rho(x - \hat{x})$) the loss becomes asymptotically negligible as $D \to 0$. Our conclusion for side information dependent distortion measures is different. At least for the case of independent $X$ and $Y$, if $m(X, Y)$ is not a function of $X$ alone, then Jensen's inequality implies

$$\lim_{D\to 0}\Big(R^{\mathrm{WZ}}(D) - R_{X|Y}(D)\Big) = \frac{1}{2}\mathbf{E}\big(\log \mathbf{E}[m(X, Y)|X]\big) - \frac{1}{2}\mathbf{E}\big[\log m(X, Y)\big] > 0.$$

This fact would not be surprising if the value of $\hat{x}$ minimizing $d(x, y, \hat{x})$ for given $x$ and $y$ depended on $y$, the side-information that is not available at the encoder in the Wyner-Ziv problem. Note, however, that the minimizing $\hat{x}$ is equal to $x$ for all $y$ and still the rate loss is positive.

## References


[1] R. Zamir, "The rate loss in the Wyner-Ziv problem," *IEEE Trans. Inform. Theory*, vol. IT-42, pp. 2073–2084, Nov. 1996.