The Multiple Description Rate Region at High Resolution

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Abstract

Consider encoding a source X into two descriptions, such that the first, the second and both descriptions allow decoding of X with distortion levels d_1, d_2 and d_0, respectively, relative to a distortion measure $\rho(x, \hat{x})$. Ozarow have found an explicit characterization for the region $\mathcal{R}^*(\sigma^2; d_1, d_2, d_0)$ of admissible rate pairs of the two descriptions, for a Gaussian source $X^* \sim \mathcal{N}(0, \sigma^2)$, relative to the squared-error distortion measure $\rho(x, \hat{x}) = (x - \hat{x})^2$. In fact, this is the only case for which the multiple description rate-distortion region is completely known. We show that for a general real valued source, a locally quadratic distortion measure of the form $\rho(x, \hat{x}) = w(x)^2 (x - \hat{x})^2 + o((x - \hat{x})^2)$, and *small* distortion levels, the region of admissible rate pairs equals approximately

$$\mathcal{R}^* \left(P_x 2^{2E\{\log w(X)\}}; d_1, d_2, d_0 \right)$$

where P_x is the entropy-power of the source. Applications to companding quantization are also considered.

Key Words: multiple descriptions, non-difference distortion measures, Shannon lower bound, high resolution.

I. Introduction and Main Result

The multiple description problem [3] arises in communication of analog source information (speech, image, video) via lossy packet networks. In this increasingly frequent scenario, a source code is broken into a few packets, some of which may not arrive to the destination. The decoder wishes to achieve a certain basic reproduction quality if a small subset of the packets arrives, and an improved quality if more packets or the whole source code arrives. Thus, portions of various size of the code should contain individually good, complementary descriptions of the source.

The basic formulation of the multiple description problem in the information theoretic literature involves two (lossless) sub-channels of rates R_1 and R_2, corresponding to two "packets", and three receivers. Each receiver corresponds to a possible case of packet arrival, the first arrived, the second arrived or both arrived. In response to a source block $\boldsymbol{x} = (x_1, \ldots x_n)$, the encoder generates two code words (indices) $f_i(\boldsymbol{x})$ and $f_2(\boldsymbol{x})$ at rates

$$R_i = \tfrac{1}{n} \log |f_i(\cdot)| \qquad i = 1, 2$$

where $|f_i|$ denotes the size of code $f_i\ (\cdot)$, and transmits codeword f_i through sub-channel $i, i = 1, 2$. The two individual ("marginal") receivers and the combined ("central") receiver then generate reconstructions $\widehat{\boldsymbol{x}}_1, \widehat{\boldsymbol{x}}_2$ and $\widehat{\boldsymbol{x}}_0$, respectively, using the decoding functions g_1, g_2 and g_0, i.e.

$$\begin{aligned}
\widehat{\boldsymbol{x}}_1 &= g_1\big(f_1(\boldsymbol{x})\big) \\
\widehat{\boldsymbol{x}}_2 &= g_2\big(f_2(\boldsymbol{x})\big) \\
\widehat{\boldsymbol{x}}_0 &= g_0\big(f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\big)\ .
\end{aligned}$$

Assume that $X_j \in \mathbb{R}^1, j = 1, \ldots, n$ is a real memoryless source, with a generating random variable X, and we wish to satisfy distortion levels d_1, d_2 and d_0 in the three non void cases of packet arrival, i.e.,

$$\frac{1}{n} \sum_{j=1}^{n} E\rho(X_j, \widehat{X}_{i,j}) \leq d_i \quad i = 0, 1, 2$$

where the reconstructions $\widehat{\boldsymbol{X}}_0, \widehat{\boldsymbol{X}}_1, \widehat{\boldsymbol{X}}_2$ are vectors in $\mathcal{R}^n$, and the distortion measure $\rho(x, \hat{x})$ is common to all receivers. Let

$$\mathcal{R}_X(d_0, d_1, d_2) = \{(R_1, R_2) : d_0, d_1, d_2 \quad \text{are achievable for some } n\} \tag{1}$$

be the set of achievable rate pairs, or inversely let

$$\mathcal{D}_X(R_1, R_2) = \{(d_0, d_1, d_2) \quad \text{are achievable with rates } (R_1 R_2)\}$$

be the set of achievable distortions for a given rate pair (R_1, R_2). The multiple description problem differs from the classical rate-distortion problem, since it is usually impossible to satisfy the Shannon limits $R_1 = R_X(d_1), R_2 = R_X(d_2)$, where $R_X(d)$ denotes the rate distortion function of the source, and at the same time obtain from the central receiver distortion d_0 which satisfies the Shannon limit $R_1 + R_2 = R_X(d_0)$.

A "single letter" characterization of the set of achievable distortions $\mathcal{D}_X(R_1, R_2)$ for the case of a Gaussian source and squared error distortion measure was found by Ozarow [7]. This set, denoted $\mathcal{D}^*(\sigma^2; R_1, R_2)$, and its inverse set $\mathcal{R}^*(\sigma^2; d_0, d_1, d_2)$, where σ^2 is the source variance, are described in the next section. For general source and distortion measure, however, the complete solution for $\mathcal{R}_X(d_0, d_1, d_2)$ is still unknown. El-Gamal and Cover [3] characterize an inner bound for $\mathcal{R}_X(d_0, d_1, d_2)$, i.e., a set of achievable rates; see Section IV. Only the special case of no excess rate sum, i.e., $R_1 + R_2 = R_X(d_0)$, was solved completely by Ahlswede [1]. For the case of a *discrete* source, Zhang and Berger [9, 10] provide inner and outer bounds which are relatively tight for the case of no excess marginal rate, i.e., $R_i = R_X(d_i), i = 1, 2$.

In a recent paper [8], the second author found inner and outer bounds on $\mathcal{R}_X(d_0, d_1, d_2)$ for general sources and the squared error distortion measure, in terms of the Gaussian rate region $\mathcal{R}^*(\cdot)$. These bounds have the form

$$\mathcal{R}^*(\sigma_x^2; d_0, d_1, d_2) \subseteq \mathcal{R}_X(d_0, d_1, d_2) \subseteq \mathcal{R}^*(P_x; d_0, d_1, d_2)\ , \tag{2}$$

and they parallel Shannon's lower and upper bounds for the rate-distortion function subject to a mean squared error constraint [2]

$$R^*(\sigma_x^2; d) \geq R_X(d) \geq R^*(P_x; d)\ , \tag{3}$$

where $\sigma_x^2, h(X)$ and $P_x = 2^{2h(X)}/2\pi e$ are the variance, the differential entropy and the entropy-power of X, and

$$R^*(\sigma^2; d) \triangleq \frac{1}{2} \log\left(\frac{\sigma^2}{d}\right) \tag{4}$$

denotes the rate-distortion function for a Gaussian source $X^* \sim \mathcal{N}(0, \sigma^2)$. Equality in (2) and (3) holds for a Gaussian source X^*, in which case $P_x = \sigma_x^2$.

As is well known, the Shannon lower bound (right hand side of (3)) is asymptotically tight for small distortion level, i.e., $R_X(d) \approx R^*(P_x; d)$ as $d \to 0$ [4]. Similarly, it is shown in [8] that the "Shannon outer bound for multiple descriptions", i.e., the right hand side of (2), is asymptotically tight for small distortion levels. (Note that these functions diverge to infinity in the limit of small distortions!)

In another recent paper [5], the first two authors generalized the concept of the asymptotic tightness of the Shannon lower bound to a class of non-difference distortion measures. Under some technical conditions, they have shown that for a locally quadratic distortion measure of the form

$$\rho(x, y) = w^2(x)(y - x)^2 + o((y - x)^2)\ , \tag{5}$$

i.e.,

$$\rho(x, y) \geq 0 \quad \text{with equality iff } y = x, \text{ and} \quad w(x)^2 = \frac{1}{2} \left.\frac{\partial^2 \rho(x, y)}{\partial y^2}\right|_{y=x}, \tag{6}$$

the rate distortion function is given asymptotically for small d by

$$\begin{aligned} R_X(d) &\approx R^*(P_x; d) + E\log w(X) \\ &= R^*\left(P_x 2^{2E\log w(X)}; d\right). \end{aligned} \tag{7}$$

Namely, at high resolution conditions the rate-distortion function associated with the weighting function $w(x)$ exceeds the quadratic rate-distortion function (i.e., that associated with $w(x) \equiv 1$) by $E\log w(X)$, or the entropy-power effectively increases by a factor of $2^{2E\log w(X)}$.

In this paper we combine the two concepts of high resolution coding discussed above, i.e., the asymptotic tightness of the Shannon lower (outer) bound, and the correction figure of a non-difference distortion measure. Our result is summarized in the following theorem.

Theorem 1 *For any real source X with entropy-power $P_x > 0$, and distortion measure ρ satisfying Linder-Zamir's conditions [5], the multiple description rate region equals asymptotically*

$$\mathcal{R}_X(d_0, d_1, d_2) \approx \mathcal{R}^*\left(P_x 2^{2E\log w(X)}; d_0, d_1, d_2\right) \tag{8}$$

where $\mathcal{R}^*(\cdot)$ is the quadratic-Gaussian multiple description rate region (see its characterization in the next section), $w^2(\cdot)$ is the quadratic Taylor coefficient of $\rho(x,y)$ defined in (6), and the approximate equality $\approx$ between the rate regions $\mathcal{R}_X$ and $\mathcal{R}^*$ means that for any given ratios $d_0/d_1 > 0$ and $d_0/d_2 > 0$, and any $\epsilon > 0$, if d_0, d_1, d_2 are small enough, then

$$\mathcal{R}^{*(-\epsilon)} \subseteq \mathcal{R}_X \subseteq \mathcal{R}^{*(+\epsilon)} \tag{9}$$

where $\mathcal{R}^{(+\epsilon)} \triangleq \{(R_1, R_2) : \ (R_1 + \epsilon, R_2 + \epsilon) \in \mathcal{R}\}$.

briefly describing Ozarow's solution for a Gaussian source in Section II.

II. Ozarow's Solution of the Gaussian Case

Ozarow [7] proved that the set of achievable distortions $\mathcal{D}^*(\sigma^2, R_1, R_2)$ for a Gaussian source with variance σ^2 is the union of all triplets d_0, d_1, d_2 satisfying

$$d_1 \geq \sigma^2 \cdot 2^{-2R_1} \tag{10a}$$

$$d_2 \geq \sigma^2 \cdot 2^{-2R_2} \tag{10b}$$

$$d_0 \geq \frac{\sigma^2 \cdot 2^{-2(R_1+R_2)}}{1 - \left(\sqrt{\pi} - \sqrt{\triangle}\right)^2} \tag{10c}$$

where $\pi = (1-d_1/\sigma^2)(1-d_2/\sigma^2)$ and $\triangle = d_1 d_2/\sigma^4 - 2^{-2(R_1+R_2)}$. In the sequel we will need also the inverse function of (10). Given a triplet of mean square errors d_0, d_1, d_2, the set $\mathcal{R}^*(\sigma^2, d_0, d_1, d_2)$ of admissible rate pairs is the union of all (R_1, R_2) satisfying

$$R_1 \geq R^*(\sigma^2; d_1) \tag{11a}$$

$$R_2 \geq R^*(\sigma^2; d_2) \tag{11b}$$

$$R_1 + R_2 \geq R^*(\sigma^2; d_1) + R^*(\sigma^2; d_2) + \delta \tag{11c}$$

where $R^*(\cdot)$ is the quadratic-Gaussian rate-distortion function (4), $\delta = \delta(\sigma^2, d_0, d_1, d_2)$ is defined by

$$\delta = \begin{cases} \frac{1}{2}\log\left(\frac{1}{1-\rho^2}\right), & d_0 \leq d_{0max} \\ 0, & d_0 \geq d_{0max} \ , \end{cases} \tag{11d}$$

$$\rho = -\frac{\sqrt{\pi\epsilon_0^2 + \gamma} - \sqrt{\pi\epsilon_0^2}}{(1-\epsilon_0)\sqrt{\epsilon_1\epsilon_2}} \ , \quad \gamma = (1-\epsilon_0)[(\epsilon_1 - \epsilon_0)(\epsilon_2 - \epsilon_0) + \epsilon_0\epsilon_1\epsilon_2 - \epsilon_0^2] \tag{11e}$$

$$\pi = (1-\epsilon_1)(1-\epsilon_2) \ , \qquad \epsilon_i = d_i/\sigma^2 \quad \text{for} \quad i = 0, 1, 2 \tag{11f}$$

and

$$d_{0max} = \frac{1}{\frac{1}{d_1} + \frac{1}{d_2} - \frac{1}{\sigma^2}} = \frac{d_1 d_2}{d_1 + d_2 - d_1 d_2/\sigma^2} \ .$$

Note that $\delta, \gamma \geq 0$ and $-1 \leq \rho \leq 0$ for all $d_1, d_2 \leq \sigma^2$ and $d_0 \leq d_{0max}$. Note also that δ depends on d_0, d_1, d_2 and σ^2 only through the ratios d_i/σ^2. As we will see in Section IV, ρ has the meaning of a correlation coefficient in the Gaussian optimal test channel. The right hand sides of (11a)-(11b) are the rate-distortion functions of the Gaussian source $X^* \sim \mathcal{N}(0, \sigma^2)$ at distortion levels d_1 and d_2 respectively. Thus, the quantity $\delta = \delta_1 + \delta_2 \geq 0$ above represents the *total excess marginal rate* (TEMR) in the Gaussian case, where $\delta_i \stackrel{\Delta}{=} R_i - \frac{1}{2}\log(\sigma^2/d_i)$, $i = 1, 2$ [1]. In the case of no excess marginal rate[2], i.e., $\delta = \triangle = 0$, we have the maximum central distortion $d_0 = d_{0max}$. As δ increases from zero to infinity (ρ varies from zero to -1), the central distortion d_0 decreases from d_{0max} to zero as $O(2^{-2\delta})$.

The case of no excess rate *sum*, i.e., $R_1 + R_2 = R_x(d_0) = \frac{1}{2}\log(\sigma^2/d_0)$ ($\pi = \Delta$ in (10)), whose solution was a breakthrough in the research of multiple descriptions [1], turns out to be not interesting practically in the Gaussian case. That is since no excess rate sum happens when

[1]We borrow the term "excess marginal rate" from [10].

[2]Both δ and Δ measure the excess of $R_1 + R_2$ over $\frac{1}{2}\log(\sigma^2/d_1) + \frac{1}{2}\log(\sigma^2/d_2)$; δ is their direct difference and Δ is their exponent difference.

$d_1 + d_2 = \sigma^2(1 + 2^{-2(R_1+R_2)}) = \sigma^2 + d_0$ (see [7]), i.e., when at least one of the two marginal receivers suffers from distortion higher than $\sigma^2/2$, meaning signal-to-noise ratio that is worse than 3dB. Having both types of excess rate (marginal-rate and rate-sum) zero is possible only in the trivial case where one of the sub-channels is disabled (e.g. $R_2 = 0$ and $d_2 = \sigma^2$).

A case of special interest for us is that of high resolution coding, i.e., the limit as d_1/σ^2 and d_2/σ^2 go to zero. In this limit the TEMR and the maximum central distortion do not depend on the variance of the source. Specifically we have

$$\begin{aligned}\delta_{HR}(d_0, d_1, d_2) &\triangleq \lim_{\sigma\to\infty} \delta(\sigma^2, d_0, d_1, d_2) \triangleq \lim_{\lambda\to 0} \delta(\sigma^2, \lambda d_0, \lambda d_1, \lambda d_2) \\ &= \frac{1}{2}\log\left(\frac{1}{1-\rho_{HR}^2}\right)\end{aligned} \tag{12}$$

where

$$\rho_{HR} = -\frac{\sqrt{d_1/d_0 - 1}\sqrt{d_2/d_0 - 1} - 1}{\sqrt{d_1 d_2/d_0^2}} \tag{13}$$

and

$$\lim_{\sigma\to\infty} d_{0max} = \frac{d_1 d_2}{d_1 + d_2}\ . \tag{14}$$

Using these definitions, the asymptotic form of the multiple description rate region $\mathcal{R}^*\left(P_x 2^{2E\log w(X)}; d_0, d_1, d_2\right)$ in the right hand side of (8) can be written as the set of all rate pairs (R_1, R_2), satisfying

$$R_1 \geq R^*(\sigma^2; d_1) + E\log w(X) \tag{15a}$$

$$R_2 \geq R^*(\sigma^2; d_2) + E\log w(X) \tag{15b}$$

$$R_1 + R_2 \geq R^*(\sigma^2; d_1) + R^*(\sigma^2; d_2) + 2E\log w(X) + \delta_{HR}\ . \tag{15c}$$

III. A Converse Theorem

In this section we derive an asymptotic outer bound for the set of achievable distortions, which establishes the converse part of Theorem 1.

Lemma 1 *Under the conditions of Theorem 1, the set of achievable distortion triplets* (d_0, d_1, d_2) *satisfies*

$$\mathcal{D}_X(R_1, R_2) \mathrel{\tilde{\subset}} \mathcal{D}^*\left(P_x 2^{2E\log w(X)}; R_1, R_2\right) \tag{16}$$

where $\mathcal{D}^*(\cdot)$ *is the quadratic-Gaussian distortion region, and the asymptotic inclusion* $\tilde{\subset}$ *is defined similarly to* $\approx$ *in (8) but with the RHS condition of (9) only.*

Note that the RHS of (16) can be written also as $\mathcal{D}^*\Big(P_x;\ R_1 - E\log w(X), R_2 - E\log w(X)\Big)$.

Proof: The proof follows the line of the proof of the converse in [7], and its generalization to a non-Gaussian source in [8], except that at few points it uses the Linder-Zamir formula [5] instead of the explicit rate distortion function, and at one point it uses a high resolution approximation to the entropy of the sum of independent random variables instead of the exact value of the entropy.

IV. A Direct Theorem

In this section we use the El-Gamal-Cover inner bound [3] to prove the direct part of our main result in Theorem 1. We recall from [3] that a distortion triplet (d_0, d_1, d_2) is achievable from two descriptions of a source X with rates R_1 and R_2 if (but not necessarily only if!) there exist random variables U and V jointly distributed with X, and three functions $\widehat{X}_1 = \widehat{X}_1(U), \widehat{X}_2 = \widehat{X}_2(V)$ and $\widehat{X}_0 = \widehat{X}_0(U, V)$, such that[3]

$$R_1 \geq I(X;U) \tag{17a}$$

$$R_2 \geq I(X;V) \tag{17b}$$

$$R_1 + R_2 \geq I(X;UV) + I(U;V) \tag{17c}$$

$$d_i \geq E\rho(X, \widehat{X}_i) \qquad i = 0, 1, 2\ . \tag{17d}$$

The El-Gamal Cover solution is optimal for a Gaussian source and square error distortion measure; the entire rate region $R^*(\sigma_x^2; d_0, d_1, d_2)$ characterized in (11) is realized in this case by choosing

$$U = X + N_1 \qquad V = X + N_2 \tag{18a}$$

$$\widehat{X}_1 = \alpha_1 U \qquad \widehat{X}_2 = \alpha_2 V \qquad \widehat{X}_0 = \beta_1 U + \beta_2 V \tag{18b}$$

where (N_1, N_2) are correlated Gaussians, independent of X, with covariance matrix

$$\mathrm{COV}(N_1, N_2) = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{19}$$

whose parameters ρ, σ_1 and σ_2 are chosen so that $\mathrm{VAR}(X|U) = d_1$, $\mathrm{VAR}(X|V) = d_2$ and $\mathrm{VAR}(X|UV) = d_0$, where $\mathrm{VAR}(A|B) \stackrel{\triangle}{=} E[A - E(A|B)]^2$ denotes conditional variance. The scaling factors $\alpha_1, \alpha_2, \beta_1$ and β_2 are chosen to yield the conditional means $\widehat{X}_1 = E(X|U), \widehat{X}_2 = E(X|V)$ and $\widehat{X}_0 = E(X|UV)$, which are linear functions since (X, U, V) are jointly Gaussian. The above implies that $\sigma_i^2 =$

[3] The characterization of the achievable rate region in (17) might be looser than that of [3], but it suffices for our purpose.

$\sigma_x^2 d_i/(\sigma_x^2 - d_i)$, $i = 1, 2$, and that ρ is given by the expression in (11e). See [7] for explicit characterization of $\alpha_1, \alpha_2, \beta_1, \beta_2$ as a function of $\sigma_x^2, d_0, d_1, d_2$.

At high resolution conditions, i.e., as $(d_0, d_1, d_2) \to 0$, we have the following limiting values for the parameters of the quadratic-Gaussian solution above:

$$\begin{array}{l} \alpha_1 = \alpha_2 = 1 \\ \beta_1 = 1 - \beta_2 \triangleq \beta_{HR} \end{array} \qquad \mathrm{COV}(N_1, N_2) = \begin{pmatrix} d_1 & \rho_{HR}\sqrt{d_1 d_2} \\ \rho_{HR}\sqrt{d_1 d_2} & d_2 \end{pmatrix} \tag{20}$$

where ρ_{HR} is given in (13), and $\beta_{HR} = (d_2 + |\rho_{HR}|\sqrt{d_1 d_2})/(d_1 + 2|\rho|\sqrt{d_1 d_2} + d_2)$ (see [7, 3]). We use this set of parameters in proving the asymptotic inner bound below.

Lemma 2 *Under the conditions of Theorem 1, the set of achievable rate pairs (R_1, R_2) satisfies*

$$\mathcal{R}_X(d_0, d_1, d_2) \mathrel{\tilde{\supset}} \mathcal{R}^*\left(P_x 2^{2E \log w(X)}; d_0, d_1, d_2\right) . \tag{21}$$

Proof To make the proof more transparent we assume the simplified form of an input weighted distortion measure $\rho(x, y) = w^2(x)(y - x)^2$. Let us substitute the following auxiliary random variables in the El-Gamal Cover inner bound (18):

$$\begin{aligned} \widehat{X}_1 &= U = X + \frac{N_1}{w(X)}, \quad \widehat{X}_2 = V = X + \frac{N_2}{w(X)}, \quad \text{and} \\ \widehat{X}_0 &= \beta_{HR} U + (1 - \beta_{HR}) V = X + \frac{\beta_{HR} N_1 + (1 - \beta_{HR}) N_2}{w(X)} \end{aligned} \tag{22}$$

where (N_1, N_2) are Gaussian with the covariance matirx in (20). It is easy to verify that that this choice of $\widehat{X}_0, \widehat{X}_1$ and $\widehat{X}_2$ satisfy the distortion constraints in (17d). Thus, ineq. (17a)-(17c) describe an inner bound for $\mathcal{R}_X(d_0, d_1, d_2)$, which includes all rate pairs (R_1, R_2) satisfying

$$R_1 \geq I\left(X; X + \frac{N_1}{w(X)}\right) \tag{23a}$$

$$R_2 \geq I\left(X; X + \frac{N_2}{w(X)}\right) \tag{23b}$$

$$R_1 + R_2 \geq I\left(X; X + \frac{N_1}{w(X)}, X + \frac{N_2}{w(X)}\right) + I\left(X + \frac{N_1}{w(X)}; X + \frac{N_2}{w(X)}\right) \tag{23c}$$

$$= I\left(X; X + \frac{N_1}{w(X)}\right) + I\left(X; X + \frac{N_2}{w(X)}\right) + I\left(\frac{N_1}{w(X)}; \frac{N_2}{w(X)} \middle| X\right) \tag{23d}$$

$$= I\left(X; X + \frac{N_1}{w(X)}\right) + I\left(X; X + \frac{N_2}{w(X)}\right) + \tfrac{1}{2}\log\left(\frac{1}{1 - \rho_{HR}^2}\right) \tag{23e}$$

where ρ_{HR} is given in (13), eq. (23d) follows using the identity $I(A; BC) = I(A; B) + I(A; C) + I(B; C|A) - I(B; C)$, and in (23e) we used the fact that $I\left(\frac{N_1}{w(X)}; \frac{N_2}{w(X)} \middle| X\right) = I(N_1; N_2)$, and then applied the formula for the mutual information between correlated Gaussians. The theorem now follows from the limit

$$I\left(X; X + \frac{N_i}{w(X)}\right) = \frac{1}{2}\log(P_x/d_i) + E\log w(X) + o(1)\ , \quad \text{as } d_i \to 0$$

shown in [5], after substituting (4) and (12), and comparing with (15).

V. Discussion: Companding Model for Multiple Description

The proof of the direct theorem in the previous section has a side benifit of directing the desing of an asymptotically optimal multiple description coding scheme. Note, first, that as $d_i \to 0$, the mutulal informations in the RHS of (23a) and (23b) satisfy

$$I\left(X; X + \frac{N_1}{w(X)}\right) \approx I(q(X); q(X) + N_i)$$

where $q(x) = \int^x w(t)dt$; see [5, 6]. Furthermore,

$$q^{-1}\Big(q(X) + N_i\Big) \approx X + N_i/w(X)\ .$$

Thus, the realization of $\mathcal{R}^*\left(P_x 2^{2E\log w(X)}; d_0, d_1, d_2\right)$ in (22) is equivalent to passing the source X through the mapping $q(\cdot)$, then applying the quadratic-Gaussian multiple description test channel of (18), and finally passing the reconstructions through the inverse mapping $q^{-1}(\cdot)$.

This information theoretic realization suggests that in practice, a combination of a compressor $q(\cdot)$, a "standard squared error" multiple description quantizer, and an expander $q^{-1}(\cdot)$, can approach the optimal performance for locally quadratic distortion measures $\rho(x, \hat{x}) \approx w^2(x)(\hat{x} - x)^2$, in the limit of high dimension and small distortion. The work of the authors [6], on entropy-coded compnding quantization for non-difference distortion measures, further supports the feasibility of such a simple modular solution.

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