Correction and Interpretation of de Buda's Theorem

Tamás Linder
Department of Telecommunications
Technical University of Budapest
1521 Stoczek u. 2, Budapest, Hungary

Christian Schlegel
Digital Communications Group
University of South Australia
The Levels, SA 5095, Australia

Kenneth Zeger
Coordinated Science Laboratory
University of Illinois
1308 W. Main St., Urbana, IL 61801

ABSTRACT

De Buda's theorem states that, for asymptotically large numbers of dimensions, there exist "structured" codes which are optimal for the AWGN channel. First, we point out an error in de Buda's proof and then we correct the proof using a slightly different approach. The original erroneous proof uses thick shells of sphere bounded lattices for its optimal codes whereas we use thin annulus lattice codes for the corrected proof. We discuss the algorithmic structure of these codes as well as the implications obtained through a coding-shaping gain argument.

SUMMARY

We correct, clarify, and interpret a recent paper [1] by R. de Buda, in which he states that there exist lattice based channel codes which meet Shannon's bound for optimal codes [2]. Unfortunately, there appears to be an error with the clever proof presented by de Buda. Here, we carefully examine de Buda's proof and discuss the problems. We show that de Buda's proof can be mended, but the resulting optimal lattice code is degenerate in the sense that its "structure" appears to be lost. More precisely, the result in [1] is valid only for lattice codes whose code points lie within a thin spherical shell. Such a code resembles more a random spherical code than a lattice code.

Shannon in [2] developed tight upper and lower bounds on the error probability of optimal codes for the AWGN channel. His random coding argument used n-dimensional codes whose M_n codewords are drawn from a uniform distribution on the surface of a sphere of radius $\sqrt{nS}$ centered at the origin. Such codes have transmission rate $R = \frac{1}{n}\log M_n$.

In [1] de Buda aimed at showing that there exist structured (namely lattice based) codes for the AWGN channel that have the same near-optimal error probability properties as Shannon's "random" codes. To this end, de Buda considers an n-dimensional lattice Λ, which is translated by a vector $\hat{s}$. The bounding region of the code is a "thick" shell (or annulus), i.e., the region T between an outer sphere and an inner sphere both centered at the origin.

De Buda's main result claims that for each dimension n, there exists a lattice code of the above type with at least 2^{nR} codepoints such that its error probability $P_e(n)$ satisfies

$$P_e(n) \leq 4F_n(\theta_b, R, S/N), \qquad (1)$$

where the right side is defined in [1]. This implies that essentially the same upper bounds are valid on the decrease of the error probability for rates below the channel's capacity as the ones Shannon derived for random codes.

There seems to be a technical error in [1] in the proof of (1), with important consequences and changes in the scope of the result. To correct the error we use a bounding region T which is more appropriately described as a *thin shell.*

Fortunately there is a way to modify de Buda's proof so that essentially all his steps remain valid. The conclusion, however will be somewhat different. The idea is to consider the code that results from the radial projection of the lattice code onto the *inner* sphere. In this way we get a code whose error probability is *larger* then that of the lattice code. Thus choosing the inner radius for each dimension n as $R_n = \sqrt{nS_n}$, where $S_n \uparrow S$ as $n \to \infty$, the above argument and de Buda's corrected result show that there exists a sequence of n-dimensional lattice codes with error probability $P_e(n)$ for which $P_e(n) \leq e^{-n[E(R,S/N)-o(1)]}$ holds, where $E(R, S/N) = \lim_n -\log F_n(\theta_b, R, S/N)$. This means that for rates satisfying $R_c < R < C$, de Buda's lattice codes have the same reliability exponent as that of optimal codes, and for rates below the critical rate R_c the error probability of these lattice codes has essentially the same exponential upper bond as Shannon's code.

The shell that contains the codepoints can no longer be be called a "thick shell"; the more appropriate description is "thin shell". It is worth noting, that since the function $F_n(\theta_b, R, S'/N)$ is continuous in S', by choosing the inner radius $S' < S$ close enough to S, de Buda's result guarantees the existence of an n-dimensional lattice code whose error probability is upper bounded by a quantity arbitrarily close to the upper bound for Shannon's code. However, the better this approximation is the less the thin shell bounded lattice code resembles a lattice code in the usual sense, and the more it looks like a "random" spherical code, for which Shannon originally proved the error bounds.

Were de Buda's original proof to be correct, one might argue that the class of sphere bounded lattice codes or even lattice bounded lattice codes are asymptotically optimal as the dimension of the signal constellation grows. However, this conclusion initially appears not to follow from our corrected version of the proof since the codepoints derived from the lattice are those which lie in a thin spherical shell, and specifically exclude the lattice points interior to the inner sphere. Adding these points to the code would invalidate our presented proof.

In effect, the radius of the thin spherical shell is made to be large enough that enough lattice points fall within the sphere as needed. The main advantages of structured codes such as those derived from lattices are generally that: (i) its points can be easily enumerated thus avoiding an exhaustive storage of the points, and (ii) signal decoding can be computed efficiently, using algorithms that exploit the lattice's structure. These advantages appear to be lost for the codes we used to correct de Buda's result.

However, an argument sustaining the asymptotic optimality of structured codes can be given using a coding/shaping gain approach. We give a discussion of this implication.

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References

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[2] C. E. Shannon, "Probability of error for optimal codes in a Gaussian channel," *Bell Syst. Tech. J.*, vol. 38, pp. 611-656, May 1959.