The capacity of some hexagonal (d, k)-constraints

Zsolt Kukorelly and Kenneth Zeger
Department of Electrical and Computer Engineering
University of California, San Diego
La Jolla, CA 92093-0407
{kukorell,zeger}@code.ucsd.edu

Abstract — **We consider two-dimensional binary codes whose bits lie on a hexagonal lattice and satisfy a (d, k) constraint in six directions. We prove that the capacity is zero if $k = d+2$ and give positive lower bounds on the capacity for $k \geq 4\lceil d/2 \rceil + 1$.**

I. Introduction and Definitions

For integers $k \geq d \geq 0$, a binary sequence *satisfies the (d, k) constraint* if two consecutive ones are separated by at least d and at most k zeros. A two-dimensional binary pattern arranged in a rectangle satisfies the (d, k) constraint if it satisfies the one-dimensional (d, k) constraint both horizontally and vertically. In general, for a two-dimensional constraint S, the *capacity of S* is defined by

$$C(S) = \lim_{n,m \to \infty} \frac{\log_2(N_S(n, m))}{mn} \qquad (1)$$

where $N_S(n, m)$ is the number of $n \times m$-rectangles satisfying S. This is asymptotically the quantity of information given by a binary digit in a two-dimensional rectangle satisfying S.

In our setting, a codeword is a function $f : \mathbb{Z}^2 \longrightarrow \{0, 1\}$, i.e., a labeling of $\mathbb{Z}^2$ with zeros and ones. A *code* is a set of codewords. A (d, k)-*constrained code* is a code whose codewords all satisfy the (d, k) constraint.

The capacity of two-dimensional (d, k) constraints has been investigated by several authors [2, 3, 5] and efficient coding methods have been proposed, especially for $(d, k) = (1, \infty)$ [4]. In all of these cases, the code bits lie on a square lattice. However, by using a hexagonal lattice, it is possible to pack more code bits per area unit. More exactly, one can place $2/\sqrt{3}$ times more code bits per area. As for square lattices, we can define a (d, k) constraint on a hexagonal lattice, in which case a 1 must be surrounded by at least d and at most k zeros in six directions.

Denote the capacity of a (d, k)-constraint for the square lattice by $C_{sq}(d, k)$ and for the hexagonal lattice by $C_{hex}(d, k)$ Thus, if $C_{hex}(d, k) \geq C_{sq}(d, k) \cdot \sqrt{3}/2$, then it is possible to place more information per area unit with a (d, k)-constrained code on the hexagonal lattice than on the square lattice.

In this paper, we investigate the zero capacity region of hexagonal (d, k) constraints, i.e., the set of pairs (d, k) with $C_{hex}(d, k) = 0$. We give a partial answer, which narrows down the unresolved cases to those d and k satisfying $d+3 \leq k \leq 2d$.

II. Main Results

A hexagonal lattice with a (d, k) constraint is topologically the same as the square lattice with a (d, k) constraint horizontally, vertically, and on the Northwest-Southeast diagonal. Any codeword that satisfies the hexagonal (d, k) constraint also satisfies the usual two-dimensional (d, k) constraint. Therefore, $C_{hex}(d, k) \geq C_{sq}(d, k)$ for all d and all k. In particular, as $C_{sq}(d, d+1) = 0$ [3], we also have $C_{hex}(d, d+1) = 0$. Also, $C_{hex}(0, 1)$ is known exactly [1] and is positive.

Theorem: $C_{hex}(d, d+2) = 0$ for all $d \geq 1$.

For $d = 1$, this is proved by showing that the pattern 101 is impossible, and thus $C_{hex}(1, 3) = C_{hex}(2, 3) = 0$. For $d \geq 2$, we show that for n, m large enough, the number of $n \times m$ rectangles that satisfy the constraint and contain the pattern 10^d1 is upper-bounded by $K2^{an+bm}$ for some constants K, a, b. Thus, the codewords with a 10^d1 pattern contribute 0 to capacity and hence $C_{hex}(d, d+2) = C_{hex}(d+1, d+2) = 0$.

Theorem: If d is a positive even integer then:

1. If $k+1$ is an odd multiple of $d+1$, then $C_{hex}(d, k) \geq \frac{1}{d+1} - \frac{1}{k+1}$;
2. if $k+1$ is an even multiple of $d+1$, then $C_{hex}(d, k) \geq \frac{1}{d+1} - \frac{1}{k+1} - \frac{d+1}{(k+1)^2}$;

This is proved by tiling the plane with a $(k+1) \times (k+1)$-square that contains free bits, i.e., points which can be labeled by 0 or by 1, such that the obtained codewords always satisfy the constraints.

Corollary:

1. For $d \geq 2$ even, $C_{hex}(d, 2d+1) > 0$.
2. For $d \geq 1$ odd, $C_{hex}(d, 2d+3) > 0$.
3. For $d \geq 2$ even, $C_{hex}(d, \infty) \geq \frac{1}{d+1}$;
4. For any $d \geq 1$, $C_{hex}(d, \infty) \geq \frac{1}{d+2}$.

References

[1] R. J. Baxter, "Hard Hexagons: Exact Solution", J. Phys. A, Vol. 13, pp. L61-L70, 1980.

[2] N. J. Calkin and H. S. Wilf, "The Number of Independent Sets in a Grid Graph", *SIAM Journal on Discrete Mathematics*, Vol. 11, pp. 54-60, February 1998.

[3] A. Kato and K. Zeger, "On the Capacity of Two-Dimensional Run Length Constrained Channels", *IEEE Trans. Inform. Theory*, Vol. 45, pp. 1527-1540, July 1999.

[4] R. Roth, P. H. Siegel, and J. K. Wolf, "Efficient Coding Schemes for the Hard-Square Model", *IEEE Trans. Inform. Theory*, to appear.

[5] W. Weeks IV and R. E. Blahut, "The Capacity and Coding Gain of Certain Checkerboard Codes", *IEEE Trans. Inform. Theory*, Vol. 42, pp. 1546-1548, September 1996.