On the Capacity of Two Dimensional Run Length Limited Codes

Akiko Kato[1]
Dept. of Mathematical Engineering and Information Physics,
University of Tokyo,
Tokyo 113-8656, JAPAN
Email kato@code.ucsd.edu

Kenneth Zeger[2]
Dept. of Electrical and Computer Engineering
University of California, San Diego,
CA 92093-0407, USA
Email zeger@ucsd.edu

Abstract — **The capacity of two dimensional run length limited codes are studied. Zero capacities are characterized, bounds on nonzero capacities are given, and encoding algorithms are also discussed.**

I. Definition

A one-dimensional binary sequence is said to satisfy a (d,k)-constraint if the number of 0's between any pair of consecutive 1's is at least d and at most k. A two-dimensional binary pattern of 0's and 1's on an $m \times n$ rectangle is said to satisfy a two-dimensional (d,k)-constraint if it satisfies a one-dimensional (d,k)-constraint both horizontally and vertically. The two-dimensional (d,k)-*capacity* is defined as

$$C_{d,k} = \lim_{m,n\to\infty} \frac{\log_2 N_{m,n}^{(d,k)}}{mn},$$

where $N_{m,n}^{(d,k)}$ is the number of patterns satisfying the two-dimensional (d,k)-constraint on an $m \times n$ rectangle.

II. Zero Capacities

It was shown in [1] that $C_{1,2} = 0$. Theorems 1 and 2 below characterize all (d,k) such that $C_{d,k} = 0$. We assume $k > d$.

Theorem 1 *For every positive d, $C_{d,d+1} = 0$.*

Theorem 2 *If $d < k$, then*

$$C_{d,k} \geq \max_{2\leq j\leq 1+\frac{k-d}{2}} \left\{ \frac{\left\lfloor 1+\frac{d}{j}\right\rfloor \log_2\left(j!\right) + \log_2(r!)}{(j+d)^2} \right\},$$

where $r = d \bmod j$.

Corollary 1 *For every positive integer d,*

$$C_{d,k} = 0 \Leftrightarrow k = d+1.$$

III. Lower Bounds

Theorems 3–5 are given in terms of the quantity $C_{1,\infty}$, which was bounded in [2] as $.587891 \leq C_{1,\infty} \leq .588339$.

Theorem 3 *For every positive integer k,*

$$C_{0,k} \geq 1 - \frac{1-C_{1,\infty}}{\lceil k/2 \rceil}.$$

Theorem 4 *If d and k are positive integers such that $\frac{k+1}{d+1}$ is an even integer, then*

$$C_{d,k} \geq \frac{1}{d+1} - \frac{2}{k+1}(1 - C_{1,\infty}).$$

Theorem 5 *For every $d \geq 2$,*

$$C_{d,\infty} \geq \frac{C_{1,\infty}}{1+\lfloor d/2 \rfloor}.$$

Theorem 6 below tightens the lower bound in Theorem 5 if and only if $d \neq 3$.

Theorem 6 *For every $d \geq 2$,*

$$C_{d,\infty} \geq \max_{1\leq s\leq d} \left\{ \frac{\left\lfloor 1+\frac{d}{s}\right\rfloor \log_2\left(s!\sum_{i=0}^{s}\binom{s}{i}\frac{1}{i!}\right) + \log_2\left(r!\sum_{i=0}^{r}\binom{r}{i}\frac{1}{i!}\right)}{(s+d)^2} \right\},$$

where $r = d \bmod s$.

IV. Upper Bounds

Theorem 7 *For every positive integer k,*

$$C_{0,k} \leq 1 - \left(\frac{1}{k+1}\right)\log_2\left(\frac{1}{1-2^{-(k+1)}}\right).$$

Theorem 8 *For every positive integer d,*

$$C_{d,\infty} \leq \frac{1}{d^2}\log_2\left(d!\sum_{i=0}^{d}\binom{d}{i}\frac{1}{i!}\right).$$

V. Asymptotic Behavior

Theorems 6 and 8 imply that $C_{d,\infty}$ decays to zero (as d grows) exactly at the rate $(\log_2 d)/d$.

Corollary 2

$$\lim_{d\to\infty}\left(\frac{d}{\log_2 d}\right)\cdot C_{d,\infty} = 1.$$

References

[1] J. J. Ashley and B. H. Marcus, "Two-Dimensional Lowpass Filtering Codes," IBM Research Division, Almaden Research Center, IBM Research Report RJ 10045 (90541), Oct. 1996.

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