Positive Capacity Region of Two-dimensional Asymmetric Run Length Constrained Channels*

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I. Introduction

Run length constraints derive from digital storage applications [2]. For nonnegative integers d and k, a binary sequence is said to satisfy a one-dimensional (d,k)-*constraint* if every run of zeros has length at least d and at most k (if two ones are adjacent in the sequence we say that a run of zeros of length zero is between them). A two-dimensional binary pattern arranged in an $m \times n$ rectangle is said to be (d_1,k_1,d_2,k_2)-*constrained* if it satisfies a one-dimensional (d_1,k_1)-constraint horizontally and a one-dimensional (d_2,k_2)-constraint vertically. The two-dimensional (d_1,k_1,d_2,k_2)-*capacity* is defined as

$$C_{d_1,k_1,d_2,k_2} = \lim_{m,n\to\infty} \frac{\log_2 N_{m,n}^{(d_1,k_1,d_2,k_2)}}{mn}$$

where $N_{m,n}^{(d_1,k_1,d_2,k_2)}$ denotes the number of $m \times n$ rectangles that are (d_1,k_1,d_2,k_2)-constrained. If $d = d_1 = d_2$ and $k = k_1 = k_2$ (this is called the *symmetric constraint*) then the two-dimensional (d_1,k_1,d_2,k_2)-capacity is called the two-dimensional (d,k)-*capacity*, and is denoted by $C_{d,k}$. A proof was given in [3] that shows the two-dimensional (d,k)-capacities exist, and essentially the same proof shows that the C_{d_1,k_1,d_2,k_2} exist.

The two-dimensional asymmetric *positive capacity region* is the set

$$\{(d_1,k_1,d_2,k_2) : C_{d_1,k_1;d_2,k_2} > 0\}.$$

A basic question is to determine which constraints actually lie in the positive capacity region and which do not. For the symmetric constraints, it was shown in [1] that $C_{1,2} = 0$ and a complete characterization of which (d,k) integer pairs yield positive capacities was given in [3] and is stated as the proposition below.

Proposition 1 *$C_{d,k} > 0$ if and only if $k - d \geq 2$ or $(d,k) = (0,1)$.*

II. Main Results

In the present paper we determine whether or not the two-dimensional capacity is positive, for a large set of asymmetric constraints (d_1,k_1,d_2,k_2), and the main results are summarized in Theorem 1. It is interesting to note that for the symmetric constraint (i.e. when $d_1 = d_2$ and $k_1 = k_2$), the capacity is zero whenever d and k are positive and differ by one, whereas for many asymmetric constraints the capacity is positive when the horizontal constraints or the vertical constraints differ by one (e.g. Theorem 1 part (ii(B)b)). However, in the asymmetric case if, for example, $k_1 = d_1 + 1 \leq d_2$ then the capacity is zero (by Theorem 1 part (i)).

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Theorem 1 *Let d_1, k_1, d_2, and k_2 be nonnegative integers such that $d_1 \leq k_1$ and $d_2 \leq k_2$. Let $d = \min(d_1,d_2)$, $D = \max(d_1,d_2)$, $k = \min(k_1,k_2)$, $K = \max(k_1,k_2)$, $\delta = k - D$, and $\Delta = K - d$. Then the following partially characterizes the positive capacity region of two-dimensional run length constrained channels:*

- (i) *If $\delta \leq 0$ then $C_{d_1,k_1;d_2,k_2} = 0$.*
- (ii) *If $\delta = 1$ then*
  - (A) *If $d = 0$ then $C_{d_1,k_1;d_2,k_2} > 0$.*
  - (B) *If $d \geq 1$ then*
    - (a) *If $\Delta \leq 1$ then $C_{d_1,k_1;d_2,k_2} = 0$.*
    - (b) *If $\Delta > d_1 = d_2$ then $C_{d_1,k_1;d_2,k_2} > 0$.*
    - (c) *If $\Delta \geq 3$ and $d = 1$ then $C_{d_1,k_1;d_2,k_2} > 0$.*
- (iii) *If $\delta \geq 2$ then $C_{d_1,k_1;d_2,k_2} > 0$.*

The only case that is presently not completely characterized in Theorem 1 is part (iiB), namely when $\delta = 1$, $d \geq 1$, and $\Delta \geq 2$. If $\delta = 1$, $d = 1$, and $\Delta = 2$ then the only capacities that need be considered are $C_{1,2,1,3}$ and $C_{1,3,2,3}$. But $C_{1,2,1,3} > 0$ from part (ii(B)b). Thus if we were able to show that $C_{1,3,2,3} > 0$ then we could replace $\Delta \geq 3$ by $\Delta \geq 2$ in part (ii(B)c). However, computer simulation suggests, but does not prove, that perhaps $C_{1,3,2,3} = 0$. This remains an open question.

Also, computer simulations suggest the plausibility of Conjecture 1 below, for which we presently do not have a proof either.

Conjecture 1 *$C_{d,d+1,d,2d} = 0$ whenever $d \geq 0$.*

Conjecture 1 would characterize with Theorem 1(ii(B)b) the positive capacity region for $k = d+1$ and $d_1 = d_2$ as:

$$C_{d,K,d,d+1} = C_{d,d+1,d,K} = 0 \text{ if and only if } K \leq 2d.$$

References

[1] J. J. Ashley and B. H. Marcus, "Two-Dimensional Lowpass Filtering Codes," IBM Research Division, Almaden Research Center, IBM Research Report RJ 10045 (90541), October 1996.

[2] K. A. Immink, P. H. Siegel, and J. K. Wolf, "Codes for Digital Recorders," *IEEE Trans. Inform. Theory*, vol. 44, pp. 2260–2299, October 1998.

[3] A. Kato and K. Zeger, "On the Capacity of Two-Dimensional Run Length Constrained Channels," *IEEE Trans. Inform. Theory*, vol. 45, no. 5, July 1999, pp. 1527–1540.