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Zero Capacity Region of Multidimensional Run Length Constraints*

Hisashi Ito
Dept. of Information Science,
Toho University,
Chiba 274-8510, Japan
his@kuro.is.sci.toho-u.ac.jp

Akiko Kato
Dept. of Math. Engineering
and Information Physics,
University of Tokyo,
Tokyo 113-8656, Japan
(Died on Feb. 27, 2000, age 32)

Zsigmond Nagy
Department of Electrical
and Computer Engineering,
University of California,
San Diego, CA 92093-0407
nagy@code.ucsd.edu

Kenneth Zeger
Department of Electrical
and Computer Engineering
University of California,
San Diego, CA 92093-0407
zeger@ucsd.edu

I. Introduction

Run length constraints derive from digital storage applications [3]. For nonnegative integers d and k, a binary sequence is (d,k)-*constrained* if there are at most k consecutive zeros and between every two ones there are at least d consecutive zeros. An n-dimensional pattern of zeros and ones arranged in an $m_1 \times m_2 \times \cdots \times m_n$ hyper-rectangle is (d,k)-*constrained* if it is (1-dimensional) (d,k)-constrained in each of the n coordinate axis directions. The n-dimensional (d,k)-*capacity* is defined as

$$C_{d,k}^{(n)} = \lim_{m_1,m_2,\ldots,m_n \to \infty} \frac{\log_2 N_{m_1,m_2,\ldots,m_n}^{(n;d,k)}}{m_1 m_2 \cdots m_n},$$

where $N_{m_1,m_2,\ldots,m_n}^{(n;d,k)}$ denotes the number of (d,k)-constrained patterns on an $m_1 \times m_2 \times \cdots \times m_n$ hyper-rectangle. A simple proof was given in [5] that shows the existence of two-dimensional (d,k)-capacities, and a slight modification of the proof can show that the n-dimensional (d,k)-capacities exist. The capacity $C_{d,k}^{(n)}$ represents the maximum number of bits of information that can be stored asymptotically per unit volume in n-dimensional space without violating the (d,k) constraint.

The study of 1-dimensional (d,k)-capacities was originally motivated by applications in magnetic storage. Interest in 2-dimensional (d,k)-capacities has recently increased due to emerging 2-dimensional optical recording devices, and the multidimensional (d,k)-capacities may play a role in future technologies as well. A tutorial on these topics is given in [3].

In general, the exact values of the various n-dimensional (d,k)-capacities are not known except in a few cases [6]. For example, in all dimensions, if $k = d$ the capacity is zero, and if $d = 0$ the capacity is positive for all $k \geq 1$. In one dimension the capacity is positive whenever $k > d \geq 0$. Very tight upper and lower bounds on the $(0,1)$-capacity were given for two dimensions in [1], improved in [2, 7], and extended to three dimensions in [7]. In [9] an encoding procedure for the 2-dimensional (d,∞)-constraint was given for all positive integer d's, and in [8] an encoding procedure for the 2-dimensional $(0,1)$-constraint was given whose coding rating comes very close to the capacity. It was shown [5] that whenever $k > d \geq 1$, the 2-dimensional capacity is zero if and only if $k = d+1$.

II. Main Results

We present two main results that characterize the zero capacity region for finite dimensions and in the limit of large dimensions. The first result generalizes the zero capacity characterization in [5] to all dimensions greater than one, which turns out to be exactly the same as in dimension 2. The second result gives a necessary and sufficient condition on d and k, such that the capacity approaches zero in the limit as the dimension n grows to infinity. These results are summarized in the following two theorems.

Theorem 1 *For every $n \geq 2$, $d \geq 1$, and $k > d$,*

$$C_{d,k}^{(n)} = 0 \Leftrightarrow k = d+1.$$

Theorem 2 *For every $d \geq 0$ and $k \geq d$,*

$$\lim_{n \to \infty} C_{d,k}^{(n)} = 0 \Leftrightarrow k \leq 2d.$$

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