Tradeoff Between Source and Channel Coding

Bertrand Hochwald
Mathematics of Communications
Bell Laboratories
Lucent Technologies, Rm. 2C-361
600 Mountain Avenue
Murray Hill, NJ 07974
e-mail: hochwald@bell-labs.com

Kenneth Zeger[1]
Department of ECE, MC 0407
University of California, San Diego
La Jolla, CA 92093-0407
e-mail: zeger@ucsd.edu

Abstract — **A problem in the transmission of analog information across a noisy discrete channel is the choice of channel code rate that optimally allocates the available transmission rate between lossy source coding and block channel coding. We establish tight bounds on the channel code rate that minimizes the average distortion of a vector quantizer cascaded with a channel coder and a binary symmetric channel. Analytic expressions are derived in the cases when the bit error probability is small and, also, when the source vector dimension is large. The optimal channel code rate is often substantially smaller than the channel capacity, and we obtain a noisy-channel version of the Zador high-resolution distortion formula.**

I. Introduction

Suppose a lossy source coder or vector quantizer takes an input vector $X \in \boldsymbol{R}^k$ and produces an m-bit output, which is expanded to n bits by a block channel coder and then sent over a binary symmetric channel. For a fixed transmission rate per source component $R = n/k$, we provide tight bounds on the best channel code rate $r = m/n$ to use. For this channel code rate, we also show that the distortion decays asymptotically (in R) as 2^{-pRr}, where the pth power of the Euclidean l_2 norm between the sent and received vectors is used as the fidelity criterion.

An m-bit vector quantizer is a function $Q(x)$ from $\boldsymbol{R}^k$ to $\{y_1, \ldots, y_M\}$, where $M = 2^m$, which partitions the space $\boldsymbol{R}^k$ into disjoint cells S_i, each of which is represented by a codevector $y_i \in \boldsymbol{R}^k$. Given a point in S_i, the quantizer outputs the m-bit index of the representative y_i. This m-bit index is then sent over a digital channel to a receiver.

When the channel is binary symmetric with error probability ϵ, the m-bit ith quantizer index changes into the jth index with probability $\epsilon^{d_{i,j}}(1-\epsilon)^{m-d_{i,j}}$, where $d_{i,j}$ is the Hamming distance between the ith and jth quantizer indices. The distortion of using Q is

$$D = \sum_{i,j=1}^{M} \epsilon^{d_{i,j}}(1-\epsilon)^{m-d_{i,j}} \int_{S_i} \|x - y_j\|^p f(x)\, dx,$$

where $f(x)$ is the source density.

Suppose we are now allowed $n > m$ channel uses to transmit the vector X. To reduce D, we may consider consider using some or all of the bits as channel coding to reduce the channel error probability. We seek the channel code rate r that minimizes the end-to-end distortion. The extreme cases $r = 0$ and $r = 1$ represent all/no channel coding.

[1]This work was supported in part by the National Science Foundation and the Joint Services Electronic Program.

II. Results

We show that for small ϵ and large R, the optimal r obeys

$$1 - \frac{2p}{k}\frac{\log\log 1/\epsilon}{\log 1/\epsilon} \lesssim r \lesssim 1 - \frac{p}{k}\frac{\log\log 1/\epsilon}{\log 1/\epsilon}, \qquad (1)$$

where logarithms are base two. On the other hand, for fixed ϵ and large k and R, the optimal r obeys

$$r \sim C - \frac{\alpha}{\sqrt{k}}, \qquad (2)$$

where

$$\alpha = \left[(2pC/\log e)[\epsilon \log^2 \epsilon + (1-\epsilon)\log^2(1-\epsilon) - (1-C)^2]\right]^{1/2},$$

and $C = 1 + \epsilon \log \epsilon + (1-\epsilon)\log(1-\epsilon)$ is the channel capacity. In both cases, the distortion D is, asymptotically in R,

$$D \sim 2^{-pRr}. \qquad (3)$$

These formulas are independent of $f(x)$.

The proofs of these results rely on balancing the exponential dependence on R of the source and channel coding components of the total distortion. The source coding exponent is given by a formula due to Zador, and the channel coding exponents include combinations of the random, expurgated, and sphere-packing exponents.

III. Interpretation

For small ϵ, the optimal channel code rate is tightly bounded in (1). This formula suggests that at least approximately $(p/k)(\log\log 1/\epsilon)/\log 1/\epsilon$ fraction, and at most approximately $(2p/k)(\log\log 1/\epsilon)/\log 1/\epsilon$ fraction, of the transmission rate R should be used for channel coding to achieve minimum distortion.

For large k, the optimal channel code rate is prescribed by (2). This formula is an asymptotic equality, rather than a bound, and implies that as the source vector dimension grows, the optimal channel code rate approaches channel capacity with rate $1/\sqrt{k}$. For small k, the optimal channel code rate can clearly be much less than the capacity.

The distortion D given in (3) is a generalization of the Zador formula $D \sim 2^{-pR}$, which is the distortion obtained over a noiseless channel. Observe from (1) that as $\epsilon \to 0$, the optimal channel code rate $r \to 1$, which is consistent with the obvious conclusion that no channel coding is needed when the channel is noiseless.