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WRAPPED SPHERICAL CODES*

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Abstract

A new class of spherical codes called *wrapped spherical codes* is constructed by "wrapping" any sphere packing Λ in Euclidean space onto a finite subset of the unit sphere in one higher dimension. The mapping preserves much of the structure of Λ, and unlike previously proposed maps, the density of wrapped spherical codes approaches the density of Λ, as the minimum distance approaches zero. In particular, wrapped spherical codes are asymptotically optimal as the minimum distance shrinks, whenever the packing Λ is optimal. Additionally, wrapped spherical codes can be effectively decoded using a decoding algorithm for Λ.

1 Introduction

A k-dimensional *spherical code* is a set of points in $\mathbb{R}^k$ that lie on the surface of a k-dimensional unit radius sphere. See [1,2] for applications of spherical codes. In this paper we concentrate on the generic spherical code design problem (with respect to minimum distance), rather than a particular application of spherical codes.

Denote the surface of the unit radius k-dimensional Euclidean sphere by

$$S_k \equiv \{\, (x_1, \ldots, x_k) \in \mathbb{R}^k : \sum_{i=1}^{k} x_i^2 = 1 \,\}, \tag{1}$$

the $(k-1)$-dimensional content (surface area) of S_k by $A_k = \frac{k\pi^{k/2}}{\Gamma(\frac{k}{2}+1)}$, and the k-dimensional content (volume) of S_k by $V_k = \frac{\pi^{k/2}}{\Gamma(\frac{k}{2}+1)}$, where Γ is the usual gamma function defined by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$. The *minimum distance* of a k-dimensional spherical code $C \subset S_k$ is defined as

$$d \equiv \min_{\substack{X,Y \in C \\ X \neq Y}} \|X - Y\|, \tag{2}$$

where $\|X - Y\|$ is the Euclidean distance in $\mathbb{R}^k$ between codepoints X and Y. The minimum distance of a spherical code is directly related to the "quality" of the code in many channel coding applications. For channel codes, one generally desires to maximize the minimum distance for a given number of codepoints.

As this paper concentrates on asymptotically small d, it is important to clarify some notation. For functions f and g, we use the notation "$f(d) = O(g(d))$" to mean that there exist positive constants c and d_0 such that $0 \leq f(d) \leq cg(d)$ for all $d \in (0, d_0)$. The dimension k shall be regarded as a constant in the asymptotic analysis.

The *angular separation* between two points $X, Y \in S_k$ is $\cos^{-1}(X \cdot Y)$. The *minimum angular separation* of spherical code C is defined as

$$\begin{aligned} \theta &\equiv 2\sin^{-1}(d/2) && (3)\\ &= d + \frac{d^3}{24} + O(d^5). && (4) \end{aligned}$$

The set of points on S_k whose angular separation from a fixed point $X \in S_k$ is less than ϕ is called a *spherical cap centered at X with angular radius ϕ* and is denoted by

$$c_X(k, \phi) \equiv \{\, Y \in S_k : X \cdot Y > \cos\phi \,\}. \tag{5}$$

When the center X of a spherical cap is not relevant, the notation may be abbreviated as $c(k, \phi)$. If two spherical caps of angular radius $\theta/2$ are centered at different codepoints of a spherical code with minimum distance d and minimum angular separation θ, then the caps are disjoint. The $(k-1)$-dimensional content of $c(k, \theta/2)$ is denoted by $A(c(k, \theta/2))$.

A *sphere packing* (or simply *packing*) is a set of mutually disjoint, equal radius, open spheres. The *packing radius* is the radius of the spheres in a packing. As defined in [3], "A packing is said to have *density* Δ if the ratio of the volume of the part of a cube covered by the spheres of the packing to the volume of the whole cube tends to the limit Δ, as the side of the cube tends to infinity." The *density Δ_C of a spherical code $C \subset S_k$ with minimum distance d* is the ratio of the total $(k-1)$-dimensional content of $|C|$ disjoint spherical caps centered at the codepoints and with angular radius $\theta/2$, to the $(k-1)$-dimensional content of S_k; that is, $\Delta_C = |C| \cdot A(c(k, \theta/2))/A_k$. Let $M(k, d)$ be the maximum cardinality of a k-dimensional spherical code with minimum distance d, and let $\Delta(k, d)$ be the maximum density among all k-dimensional spherical codes with minimum distance d. Then,

$$\Delta(k, d) \equiv \frac{M(k, d) A(c(k, \theta/2))}{A_k}. \tag{6}$$

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The value of $M(k,d)$ is easy to compute for all d when $k = 2$. However, $M(k,d)$ is unknown for all $k \geq 3$ except for a handful of values of d, although a number of bounds have been given [4–11]. For asymptotically small d, the tightest known upper bounds on $M(k,d)$ are given in [7] for $k = 3$ and in [4] for $k \geq 4$, and code constructions in [10, 12, 13] provide the tightest known lower bound. However, in several dimensions, there exists a nonvanishing gap between these upper and lower bounds as $d \to 0$.

A family of codes $\{C(k,d)\}$ is *asymptotically optimal* if $|C(k,d)|/M(k,d) \to 1$ as $d \to 0$, or equivalently, if $\Delta_{C(k,d)}/\Delta(k,d) \to 1$ as $d \to 0$. It has been shown that $\lim_{d\to 0} \Delta_{C(k,d)}$ is equal to the density of the densest $(k-1)$-dimensional sphere packing in [1]. Given this packing, we show that this limit can always be achieved using our construction. Hence, given a densest packing in $\mathbb{R}^{k-1}$, we construct asymptotically optimal spherical codes. Figure 1 shows bounds on the asymptotic densities of the best spherical codes for up to 49 dimensions.

2 Known lower bound constructions

Any known k-dimensional spherical code with minimum distance d gives a lower bound on $M(k,d)$, and hence on $\Delta(k,d)$. Much work has been done to find the best spherical codes, such as from binary codes [14], shells of lattices [15], permutations of a set of initial vectors [16], simulated annealing or repulsion-energy methods [10,17], concatenations of lower dimensional codes [11], projections of lower dimensional objects [6, 8, 10], and other means [18,19]. For more comprehensive references, see [1,2].

Unfortunately, none of the spherical coding methods above performs well in a fixed dimension k, as $d \to 0$. Also, many of the methods above produce spherical codes for only a finite number of minimum distances d. Recent developments have shown that it is possible to obtain asymptotically optimal k-dimensional spherical codes whenever the laminated lattice is the densest $(k-1)$-dimensional sphere packing [2, 12, 13]. This paper improves upon this result by providing a construction method which directly maps any $(k-1)$-dimensional packing onto a finite subset of S_k. It also allows efficient decoding to be performed by using a few simple operations in conjunction with the best decoding algorithm for the underlying packing.

3 Wrapped spherical codes

Any spherical code can be described by the projection of its codepoints to the interior of a sphere of one less dimension via the mapping $\left(x_1, \dots, x_{k-1}, \sqrt{1-\sum_{i=1}^{k-1} x_i^2}\right) \to (x_1, \dots, x_{k-1})$. Conversely, a k-dimensional spherical code may be obtained by placing codepoints within S_{k-1} and projecting each codepoint onto S_k using the reverse mapping. This simple mapping was used by Yaglom [6] to map a $(k-1)$-dimensional lattice Λ onto S_k. However, the distortion created by mapping Λ to S_k gives poor asymptotic spherical code densities, even if Λ is the densest lattice in $k-1$ dimensions, as summarized in Figure 1. This is due to the "warping" effect on the codepoints near the boundary.

In this section we introduce a new mapping which results in less distortion of the original lattice. The mapping effectively "wraps" any packing in $\mathbb{R}^{k-1}$ around S_k (actually into a finite subset of S_k), and hence we refer to the spherical codes it constructs as wrapped spherical codes. This technique creates codes of any size and thus provides a lower bound on achievable minimum distance as a function of code size. We shall show that the spherical code density approaches the density of the underlying packing, as $d \to 0$.

3.1 Construction of wrapped spherical codes

Let Λ be a sphere packing in $\mathbb{R}^{k-1}$ with minimum distance d and density Δ_Λ. Λ may be either a lattice packing or a nonlattice packing. Let $0 = \xi_0 < \cdots < \xi_N = 1$, and for $x \in [0,1]$, let $\underline{\xi}(x) \equiv \max\{\xi_i : \xi_i \leq x\}$ and $\overline{\xi}(x) \equiv \min\{\xi_i : \xi_i > x\}$. The real numbers $\xi_0, \dots, \xi_N$ are referred to as *latitudes* and will be chosen later to yield a large code size. The ith *annulus* is defined as the set of points $(x_1, \dots, x_k) \in S_k$ that satisfy $\xi_i \leq x_k < \xi_{i+1}$ (i.e., points between consecutive latitudes). Define the many-to-one function $f' \colon S_k \to \mathbb{R}^{k-1}$ by

$$f'(x_1, \dots, x_k) = g(x_k) \cdot \frac{(x_1, \dots, x_{k-1})}{\sqrt{1-x_k^2}}, \tag{7}$$

where

$$\begin{aligned} g(x) &= \Bigg(\sqrt{1-\underline{\xi}(x)^2} - \\ &\quad \sqrt{(|x|-\underline{\xi}(x))^2 + \left(\sqrt{1-\underline{\xi}(x)^2} - \sqrt{1-x^2}\right)^2}\Bigg)_+, \end{aligned}$$

and $(x)_+ = \max(0,x)$. If $X = (x_1, \dots, x_k)$ and $Y = f'(X) \neq 0$, then

$$\begin{aligned} \sqrt{1-\underline{\xi}(x_k)^2} - \|Y\| &= \Bigg\| \left(\sqrt{1-\underline{\xi}(x_k)^2}, \underline{\xi}(x_k)\right) - \\ &\qquad \left(\sqrt{1-x_k^2}, |x_k|\right) \Bigg\|, \end{aligned}$$

which is shown geometrically in Figure 2. Define the *buffer region* as the set

$$\begin{aligned} B' &= \Bigg\{ (x_1, \dots, x_k) \in S_k : (|x_k| - \underline{\xi}(x_k))^2 + \\ &\qquad \left(\sqrt{1-\underline{\xi}(x_k)^2} - \sqrt{1-x_k^2}\right)^2 < d^2 \Bigg\}. \end{aligned}$$

A useful spherical code with respect to Λ is defined by $\hat{C}^\Lambda = (f')^{-1}(\Lambda \setminus \{0\}) \setminus B'$. Figure 3 illustrates this three

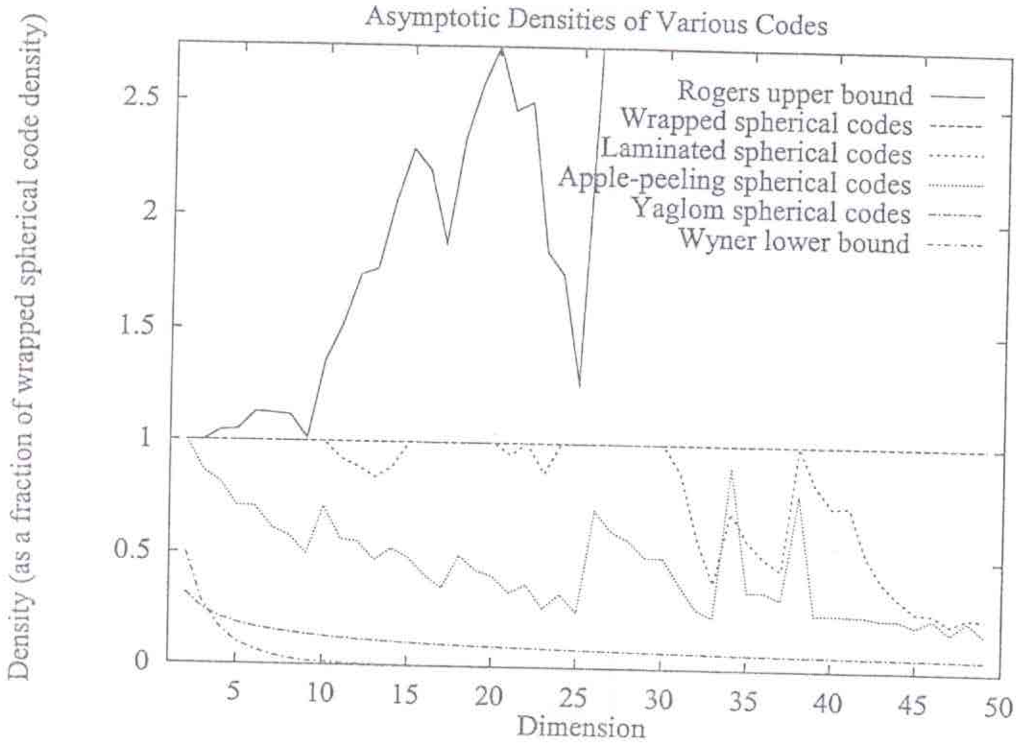

Figure 1: Comparison of the asymptotic density of various spherical codes versus the density of wrapped spherical codes. The wrapped codes were constructed with respect to the densest known packing in each dimension.

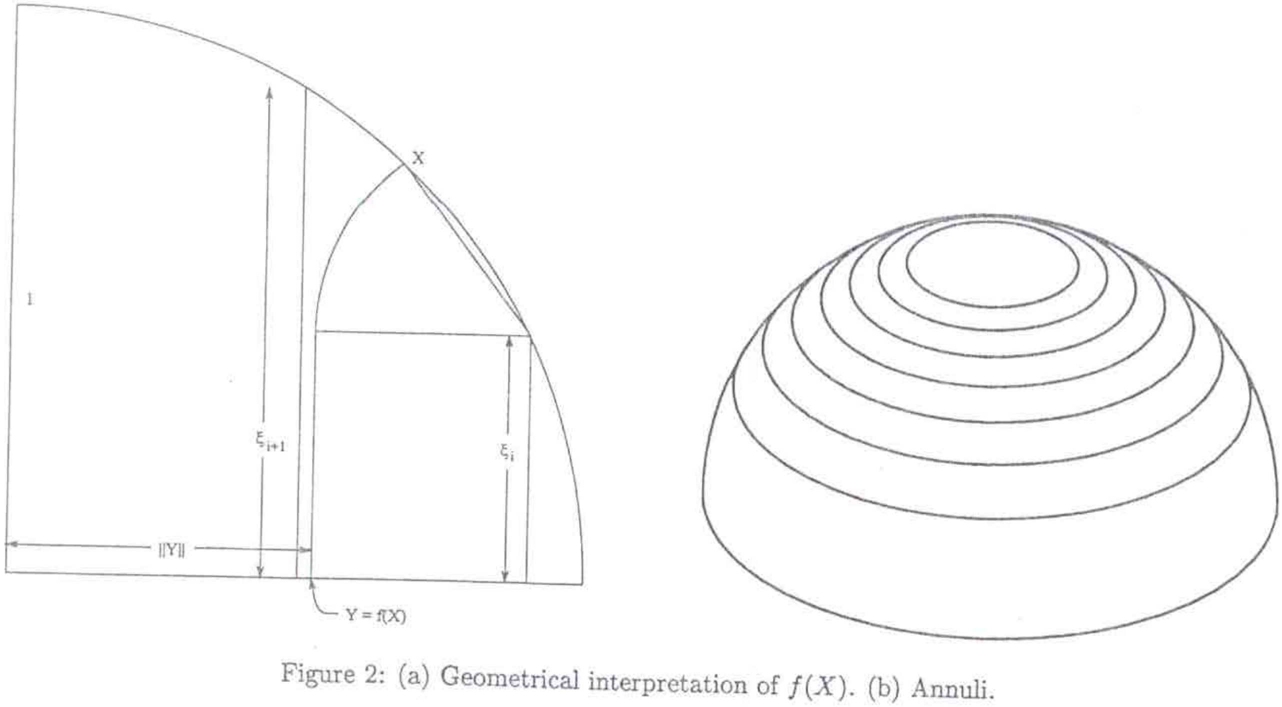

Figure 2: (a) Geometrical interpretation of $f(X)$. (b) Annuli.

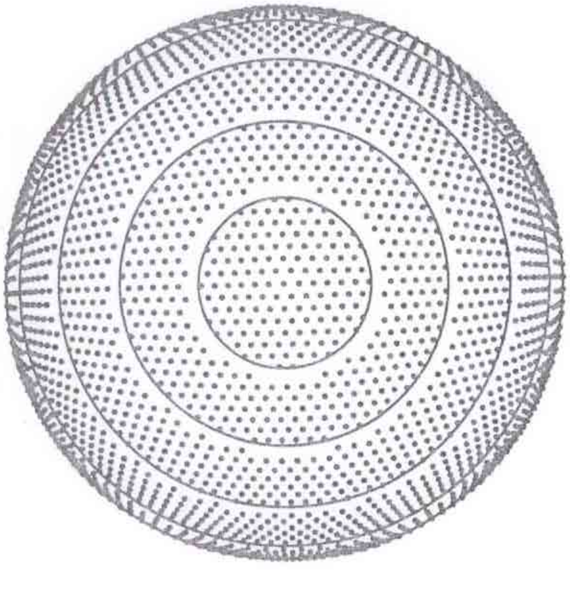

Figure 3: Perspective view of a wrapped spherical code with respect to the hexagonal lattice.

dimensional spherical code with respect to the hexagonal lattice with minimum distance 0.05.

Let

$$f(X) = \begin{cases} f'(x_1, \ldots, x_{k-2}, x_{k-1}, x_k) & \text{if } |x_k|^2 \leq 1/2 \\ f'(x_1, \ldots, x_{k-2}, x_k, x_{k-1}) & \text{if } |x_k|^2 > 1/2 \end{cases},$$

and let

$$B = B' \cup \left\{ (x_1, \ldots, x_k) \in S_k : \left(|x_k| - \frac{1}{\sqrt{2}} \right)^2 + \left(\frac{1}{\sqrt{2}} - \sqrt{1 - x_k^2} \right)^2 < d^2 \right\}.$$

The *wrapped spherical code* with respect to a packing Λ having minimum distance d is defined by

$$C^\Lambda = f^{-1}(\Lambda \setminus \{0\}) \setminus B.$$

Note that C^Λ depends on those latitudes ξ_i which satisfy $\xi_i \leq 1/\sqrt{2}$. Geometrically, C^Λ is identical to $\hat{C}^\Lambda$ for points whose last coordinate has magnitude at most $1/\sqrt{2}$, and is a rotation of $\hat{C}^\Lambda$ by $\pi/2$ for the remaining points. In the following two subsections, we show that C^Λ has good asymptotic density properties and has an efficient decoding algorithm.

As this paper is chiefly concerned with asymptotic performance, we will not concentrate on small codebook improvements possible for moderately large minimum distances. A number of simple improvements are possible. One such improvement involves the buffer regions B' and B, which are included in the code definitions solely to insure the minimum distance requirement is met. For a particular value of d, a careful choice of a latitudes $\{\xi_i\}$ may make much of the buffer region unnecessary.

The inverse mapping $(f')^{-1}$ may be computed using the following lemma.

Lemma 1 *For every $Y \in \mathbb{R}^{k-1} \setminus \{0\}$,*

$$(f')^{-1}(Y) = \left\{ \frac{g_i Y}{||Y||} \pm (0, \ldots, 0, \sqrt{1 - g_i^2}) : 0 \leq h_i < \sqrt{(\xi_{i+1} - \xi_i)^2 + \left(\sqrt{1 - \xi_i^2} - \sqrt{1 - \xi_{i+1}^2} \right)^2} \right\},$$

where $h_i = \sqrt{1 - \xi_i^2} - ||Y||$ and $g_i = \left(1 - \frac{h_i^2}{2}\right) \sqrt{1 - \xi_i^2} - \frac{h_i \xi_i}{2} \sqrt{4 - h_i^2}$.

Proof: Omitted.

Lemma 1 also allows f^{-1} to be calculated, via

$$f^{-1}(Y) = \left\{ (x_1, \ldots, x_k) \in (f')^{-1}(Y) : x_k \leq 1/\sqrt{2} \right\} \cup \left\{ (x_1, \ldots, x_{k-2}, x_k, x_{k-1}) \in (f')^{-1}(Y) : x_k > 1/\sqrt{2} \right\}.$$

The image under f' of an annulus in S_k is a region bounded by two concentric $(k-1)$-dimensional spheres in $\mathbb{R}^{k-1}$.

Lemma 2 *If $X = (x_1, \ldots, x_k) \in S_k$ and $Y = (y_1, \ldots, y_k) \in S_k$ belong to the same annulus of $\hat{C}^\Lambda$, then*

$$||f'(X) - f'(Y)||^2 \leq ||X - Y||^2.$$

If, additionally, $\xi_i = \sin\left(i\sqrt{d}\right)$, $x_k, y_k \leq 1/\sqrt{2}$, and $||f'(X) - f'(Y)|| \leq d$, then

$$||X - Y||^2 - 3d^{5/2} + O(d^3) \leq ||f'(X) - f'(Y)||^2.$$

Proof: Omitted.

Note that if $\xi_i = 1/\sqrt{2}$ for some i, then Lemma 2 also holds when f' is replaced by f.

Corollary 1 *If Λ is a sphere packing with minimum distance d, then the minimum distance of the wrapped spherical code C^Λ is also d.*

Proof: If distinct $X, Y \in C^\Lambda$ belong to the same annulus, then $||X - Y|| \geq ||f(X) - f(Y)|| \geq d$, since the minimum distance of Λ is d. If X and Y belong to different annuli, then the definition of B guarantees their separation is d. ■

3.2 Asymptotic density of the wrapped spherical code

Let $\{\xi_i^{(d)}\}$ be the partition of $[0, 1/\sqrt{2}]$ used in the definition of a wrapped spherical code C^Λ that has minimum distance d. Let $\phi_i = \sin^{-1} \xi_{i+1}^{(d)} - \sin^{-1} \xi_i^{(d)}$ denote the *angular separation* of the ith annulus. We show that if the maximum angular separation between annuli, $\overline{\phi} \equiv \max_i \phi_i$ approaches 0 as $d \to 0$ and the minimum angular separation $\underline{\phi} \equiv \min_i \phi_i$ does not approach zero too quickly, then the density of the wrapped code approaches the density of Λ.

Theorem 1 *Let Λ be a $(k-1)$-dimensional sphere packing with minimum distance d. Let C^Λ be a wrapped spherical code with respect to Λ and with latitudes $\xi_1, \cdots, \xi_N$. If the maximum and minimum annulus angular separations satisfy $\lim_{d \to 0}[\overline{\phi} + (d/\underline{\phi})] = 0$, then the asymptotic density of C^Λ approaches the density of Λ, i.e., $\lim_{d \to 0} \Delta_{C^\Lambda} = \Delta_\Lambda$.*

Proof: Omitted.

One line in the proof implies the following corollary.

Corollary 2 *Let Λ be a $(k-1)$-dimensional sphere packing with minimum distance d, and let C^Λ be a wrapped spherical code with respect to Λ and with latitudes given by $\xi_i = \sin\left(i\sqrt{d}\right)$ for $0 \le i \le \pi/(2\sqrt{d})$. Then the spherical code density satisfies $|\Delta_{C^\Lambda} - \Delta_\Lambda| \le O(\sqrt{d})$.*

3.3 Decoding wrapped spherical codes

An important question in channel decoding and quantization encoding is how to efficiently find the nearest codepoint to an arbitrary point in $\mathbb{R}^k$ (see, e.g., [18]). Often, an advantage of a structured code is that codepoints themselves need not be stored explicitly.

If the k-dimensional signal $X \in C^\Lambda$ is sent across an additive white Gaussian noise (AWGN) channel, then the received signal is $R = X + N$, where N is a zero-mean Gaussian random vector with variance σ^2. The maximum likelihood decoder is a minimum distance decoder, i.e., given R, the decoder output is $\tilde{X} = \arg\min_{X \in C^\Lambda} \|X - R\|$, the closest codepoint to R. For any $R \in \mathbb{R}^k$ and any spherical code $C(k, d)$, the nearest codepoint of $C(k, d)$ to R is the same as the nearest codepoint of $C(k, d)$ to $R/\|R\|$. Hence, in the following, we assume $R \in S_k$.

We now evaluate the performance of an efficient suboptimal decoding method. Given a received vector $R \in S_k$, let the decoder output be

$$\hat{X} = \arg\min_{X \in C^\Lambda} \|f(X) - f(R)\|.$$

Note that $X \in C^\Lambda$ implies $f(X) \in \Lambda$. Let Y be a nearest neighbor of $f(R)$ in Λ. There is at most one candidate in the set $f^{-1}(Y)$ which could be a nearest neighbor to R, namely, the element E which is in the same annulus as R. However, because of the buffer region B, E might not be in C^Λ. This happens with probability $O(\sqrt{d})$ or less, for B covers $O(\sqrt{d})$ of the sphere. (Such an E exists provided $\|Y\| \le 1$ and R is not within d of the border of an annulus, which holds with probability $1 - O(\sqrt{d})$.) Thus, with probability $1 - O(\sqrt{d})$,

$$\hat{X} \in f^{-1}\left(\arg\min_{Y \in \Lambda} \|Y - f(R)\|\right),$$

which involves only f, f^{-1}, and the decoding algorithm for Λ.

It is known that when points from the packing Λ with minimum distance d are used on an AWGN channel, the probability of symbol error is $\tau Q(\frac{d}{2\sigma})$ (see, e.g., [20]), where τ is the average number of codepoints at distance $2d$ from a codepoint and where Q is the complementary error function defined by $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-x^2/2} dx$. The following theorem shows that the performance of efficiently decoding C^Λ is asymptotically close to the performance of Λ.

Theorem 2 *Let Λ be a $(k-1)$-dimensional packing with minimum distance d, and let C^Λ be a wrapped spherical code with respect to Λ and with latitudes $\xi_i = \sin(i\sqrt{d})$. Let P_e be the probability of symbol error when C^Λ is used on an AWGN channel with equiprobable inputs and the decoder output is $\hat{X} = \arg\min_{X \in C^\Lambda} \|f(X) - f(R)\|$. Then $P_e \le \tau Q\left(\frac{d}{2\sigma}(1 - O(d^{1/4}))\right)$.*

Proof: Omitted.

4 Conclusions

A new technique was presented that constructs wrapped spherical codes in any dimension and with any minimum distance. The construction is performed by defining a map from $\mathbb{R}^{k-1}$ to S_k. Although any set of points in $\mathbb{R}^{k-1}$ may be wrapped to S_k using our technique, if the densest packing in $\mathbb{R}^{k-1}$ is used the wrapped spherical codes are asymptotically optimal, in the sense that the ratio of the minimum distance of the constructed code to the upper bound approaches one as the number of codepoints increases. This demonstrates the tightness of the upper bound in [7], asymptotically, and that previous lower bounds are not asymptotically optimal.

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References

[1] J. Hamkins and K. Zeger, "Asymptotically efficient spherical codes—Part I: Wrapped spherical codes." Submitted to IEEE Trans. Inform. Theory, Dec. 1995.

[2] J. Hamkins and K. Zeger, "Asymptotically efficient spherical codes—Part II: Laminated spherical codes." Submitted to IEEE Trans. Inform. Theory, Dec. 1995.

[3] C. A. Rogers, "The packing of equal spheres," *Proceedings of the London Mathematical Society*, vol. 8, pp. 609–620, 1958.

[4] H. S. M. Coxeter, *Twelve Geometric Essays*. Southern Illinois University Press, 1968.

[5] R. A. Rankin, "The closest packing of spherical caps in n dimensions," *Proc. Glasgow Math. Assoc.*, vol. 2, pp. 139–144, 1955.

[6] I. M. Yaglom, "Some results concerning distributions in n-dimensional space." Appendix to Russian edition of Fejes Toth's *Lagerungen in der Ebene, auf der Kugel und in Raum*, 1958.

[7] L. Fejes Tóth, "Kugelunterdeckungen und Kugelüberdeckungen in Räumen konstanter Krümmung," *Archiv Math.*, vol. 10, pp. 307–313, 1959.

[8] A. Wyner, "Capabilities of bounded discrepancy decoding," *The Bell System Technical Journal*, vol. 44, pp. 1061–1122, July-August 1965.

[9] G. A. Kabatyanskii and V. I. Levenšteĭn, "Bounds for packings on a sphere and in space (English translation)," *Problemy Peredachi Informatsii*, vol. 14, no. 1, pp. 3–25, 1978.

[10] A. A. E. Gamal, L. A. Hemachandra, I. Shperling, and V. K. Wei, "Using simulated annealing to design good codes," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 116–123, Jan. 1987.

[11] Z. Yu, "On spherical codes and coded modulation," Master's thesis, University of Hawaii at Hanoa, 1992.

[12] J. Hamkins and K. Zeger, "Asymptotically optimal spherical codes," in *Proc. CISS*, pp. 52–57, Mar. 1995.

[13] J. Hamkins and K. Zeger, "Asymptotically optimal spherical codes," in *Proc. ISIT*, p. 184, Sept. 1995.

[14] T. Ericson and V. Zinoviev, "Spherical codes generated by binary partitionings of symmetric pointsets," *IEEE Trans. Inform. Theory*, vol. 41, pp. 107–129, Jan. 1995.

[15] N. J. A. Sloane, "Tables of sphere packings and spherical codes," *IEEE Trans. Inform. Theory*, vol. IT-27, pp. 327–338, May 1981.

[16] D. Slepian, "Permutation modulation," *Proceedings of the IEEE*, vol. 53, pp. 228–236, Mar. 1965.

[17] D. A. Kottwitz, "The densest packing of equal circles on a sphere," *Acta Crystallography*, vol. A47, pp. 158–165, 1991.

[18] J. Gao, L. D. Rudolph, and C. R. P. Hartmann, "Iteratively maximum likelihood decodable spherical codes and a method for their construction," *IEEE Trans. Inform. Theory*, vol. 34, pp. 480–485, May 1988.

[19] J. K. Karlof, "Decoding spherical codes for the Guassian channel," *IEEE Trans. Inform. Theory*, vol. 39, pp. 60–65, Jan. 1993.

[20] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices, and Groups*. Springer-Verlag, 1993.