Quantization of Multiple Sources Using Integer Bit Allocation *

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Abstract

Asymptotically optimal bit allocation among a set of quantizers for a finite collection of sources was determined in 1963 by Huang and Schultheiss. Their solution, however, gives a real-valued bit allocation, whereas in practice, integer-valued bit allocations are needed. We compare the performance of the Huang-Schultheiss solution to that of an optimal integer-valued bit allocation. Specifically, we derive upper and lower bounds on the deviation of the mean squared error using optimal integer-valued bit allocation from the mean squared error using optimal real-valued bit allocation. One consequence shown is that optimal integer-valued bit allocations do not necessarily achieve the same performance as that predicted by Huang-Schultheiss, for asymptotically large transmission rates. We also prove that integer bit allocation vectors that minimize the Euclidean distance to the optimal real-valued bit allocation vector are optimal integer bit allocations.

1. Introduction

The quantizer bit allocation problem is to determine the individual rates of a finite collection of quantizers so as to minimize the sum of their distortions, subject to a constraint on the sum of their rates. Bit allocation arises in applications such as speech, image, and video coding.

Huang and Schultheiss [3] solved the bit allocation problem when the mean squared error of each quantizer decreases exponentially as its rate grows. Segall [4] generalized [3] by finding optimal real valued bit allocations when the mean squared error of each quantizer is a convex function of its rate.

All of the papers cited above have allowed arbitrary real-valued bit allocations. Real applications impose integer-value constraints on the bits used. In practice, sometimes the real-valued Huang-Schultheiss bit allocation is rounded to the nearest integer, in a manner that does not violate the overall bit budget. Alternatively, searches of integer-valued bit allocations near the Huang-Schultheiss solution are performed and the best such allocation that satisfies the bit budget is used. There are also many examples of algorithmic techniques

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for obtaining good integer-valued bit allocations (we do not cite them here due to limited space).

In this paper we provide some theoretical analysis of optimal integer bit allocations, by comparing the performance of the Huang-Schultheiss solution for a real-valued bit allocation to that of an optimal integer-valued bit allocation. Specifically, we derive upper and lower bounds on the deviation of the mean squared error using optimal integer-valued bit allocation from the mean squared error using optimal real-valued bit allocation.

2. Preliminaries

Let $X_1, \ldots, X_k$ be scalar sources with positive variances $\sigma_1^2, \ldots, \sigma_k^2$. These sources are scalar quantized with resolutions $B_1, \ldots, B_k$, measured in bits. Throughout this paper, we assume $k \geq 2$. The goal in bit allocation is to determine the values of the quantizer resolutions, subject to a constraint on their sum, so as to minimize the (possibly weighted) sum of the resulting mean squared errors of the k quantizers.

Let $\mathbb{R}$ denote the reals and $\mathbb{Z}$ denote the integers. Also, let

$$
\begin{aligned}
B &= (B_1, \ldots, B_k) \\
|U| &= \sum_{i=1}^{k} U_i \quad \forall U \in \mathbb{R}^k \\
g &= \left(\prod_{i=1}^{k} \sigma_i^2 \right)^{1/k} \\
\mathcal{F}_R(b) &= \{U \in \mathbb{R}^k : |U| = b\} && (b \geq 0) \\
\mathcal{F}_I(b) &= \{U \in \mathbb{Z}^k : |U| = b\} && (b \geq 0).
\end{aligned}
$$

The vector B will be called the *bit allocation* and the scalar b the *bit budget*. $\mathcal{F}_R(b)$ and $\mathcal{F}_I(b)$ are, respectively, the sets of all real and integer valued bit allocations B with bit budgets b. Bit allocations in $\mathcal{F}_I(b)$ are said to be *integral*. We use the notation $x = b \bmod k$ to mean that $k \mid (b - x)$ and $0 \leq x \leq k - 1$.

Assume the mean squared error of the ith quantizer is

$$d_i = h_i \sigma_i^2 4^{-B_i} \tag{1}$$

where $h_i = \left(\int |f_{X_i/\sigma_i}|^{1/3} \right)^3 / 12$ and f_{X_i} denotes the probability density function of X_i. It is known that (1) is satisfied for asymptotically optimal scalar quantization [2]. Assume, without loss of generality[1], that $h_i = h$ for all i. The total mean squared error (MSE) resulting from B is

$$d = \sum_{i=1}^{k} d_i. \tag{2}$$

Note that d is a function of B.

[1] Generalizing our results to different h_i is straightforward.

For each integer $b \geq 1$, let

$$B^{or} = \operatorname*{argmin}_{B \in \mathcal{F}_R(b)} \sum_{i=1}^{k} h\sigma_i^2 4^{-B_i}$$
$$d^{or} = \sum_{i=1}^{k} h\sigma_i^2 4^{-B_i^{or}}.$$

In 1963, Huang and Schultheiss [3] derived the optimal high resolution real-valued bit allocation for the multiple source quantization problem. Their result, stated in the following lemma, shows, in particular, that B^{or} is unique. We call B^{or} the *optimal real-valued bit allocation* and d^{or} the *MSE* achieved by B^{or}.

Lemma 2.1. *For each integer* $b \geq 1$,

$$B^{or} = \frac{b}{k}(\underbrace{1, \ldots, 1}_{k}) + \frac{1}{2}\left(\log_2 \frac{\sigma_1^2}{g}, \ldots, \log_2 \frac{\sigma_k^2}{g}\right)$$
$$d^{or} = khg4^{-b/k}.$$

Lemma 2.1 implies that the components of the bit allocation B^{or} are positive for sufficiently large bit budget b, but B^{or} is not necessarily integral.

For any bit allocation $B \in \mathcal{F}_R(b)$, we have

$$d = \sum_{i=1}^{k} h\sigma_i^2 4^{-B_i} \tag{3}$$
$$= h4^{-b/k} \sum_{i=1}^{k} \sigma_i^2 \frac{g}{\sigma_i^2} 4^{(B^{or}-B)_i} \tag{4}$$
$$= hg4^{-b/k} \cdot \sum_{i=1}^{k} 4^{(B^{or}-B)_i} \tag{5}$$
$$\geq k \left(\prod_{i=1}^{k} 4^{(B^{or}-B)_i}\right)^{1/k} \cdot hg4^{-b/k} \tag{6}$$
$$= khg4^{-b/k}$$
$$= d^{or}$$

where (3) follows from (1) and (2); (4) follows from Lemma 2.1; and (6) follows from the arithmetic-geometric mean inequality.

For any k scalar sources and for each integer $b \geq 1$, let

$$d^{oi} = \inf_{B \in \mathcal{F}_I(b)} \sum_{i=1}^{k} h\sigma_i^2 4^{-B_i}$$
$$\mathcal{Q}_{oi} = \left\{ B \in \mathcal{F}_I(b) : \sum_{i=1}^{k} h\sigma_i^2 4^{-B_i} = d^{oi} \right\}.$$

These equations are equivalent to

$$\begin{aligned} d^{oi} &= \min_{B \in \mathcal{F}_I(b)} hg4^{-b/k} \sum_{i=1}^{k} 4^{(B^{or}-B)_i} \\ \mathcal{Q}_{oi} &= \left\{ B \in \mathcal{F}_I(b) : hg4^{-b/k} \sum_{i=1}^{k} 4^{(B^{or}-B)_i} = d^{oi} \right\}. \end{aligned}$$

We call $\mathcal{Q}_{oi}$ the set of *optimal integer bit allocations*. From (5), the quantity d^{oi} is the MSE resulting from any $B \in \mathcal{Q}_{oi}$.

2.1. Lattice Tools

We exploit certain facts from lattice theory to establish bit allocation results, specifically Theorems 5.1, and 6.1. In particular, the lattice A_{k-1} is useful for analyzing bit allocations for k scalar sources since it consists of points with k integer coordinates which sum to zero. Most of the following definitions and notation are adapted from [1].

For $k \geq 1$, the A_k lattice is defined as

$$A_k = \{U \in \mathbb{Z}^{k+1} : |U| = 0\}.$$

For the lattice A_k, the *Voronoi cell* $V(Y)$ associated with any point $Y \in A_k$ is the set

$$V(Y) = \{U \in \mathbb{R}^{k+1} : \|U - Y\| \leq \|U - W\|, \ \ \forall W \in A_k\}.$$

Let

$$\mathcal{H}^k = \left\{U \in \mathbb{R}^{k+1} : |U| = 0\right\}.$$

The lattice A_k is a subset of $\mathbb{R}^{k+1}$ and also a subset of the k-dimensional hyperplane $\mathcal{H}^k$. Define the quantity

$$V_k(Y) = V(Y) \cap \mathcal{H}^k.$$

For $0 \leq j \leq k$, define

$$c(k,j) = \frac{1}{k+1}(\underbrace{-j, \ldots, -j}_{k+1-j}, \underbrace{k+1-j, \ldots, k+1-j}_{j}). \tag{7}$$

3. Closest Integer Bit Allocation

For any k scalar sources and for each integer $b \geq 1$, let

$$\begin{aligned} \mathcal{Q}_{ci} &= \left\{ B \in \mathcal{F}_I(b) : \|B - B^{or}\| = \min_{\hat{B} \in \mathcal{F}_I(b)} \|\hat{B} - B^{or}\| \right\} \\ \mathcal{D}_{ci} &= \left\{ \sum_{i=1}^{k} h\sigma_i^2 4^{-B_i} : B \in \mathcal{Q}_{ci} \right\} \\ \Delta &= \{B - B^{or} : B \in \mathcal{Q}_{ci}\}. \end{aligned}$$

Note that Δ is a function of $\sigma_i^2, \ldots, \sigma_k^2$ and b, although we will notationally omit these dependencies. The definition of the set $\mathcal{D}_{ci}$ is equivalent to

$$\mathcal{D}_{ci} = \left\{ hg4^{-b/k} \sum_{i=1}^{k} 4^{(B^{or}-B)_i} : B \in \mathcal{Q}_{ci} \right\}.$$

$\mathcal{Q}_{ci}$ is the set of closest integer bit allocations, with respect to Euclidean distance, to the optimal real-valued bit allocation. Note that each $B \in \mathcal{Q}_{ci}$ is, in general, different from a bit allocation obtained by finding the closest integer to each component of B^{or}, since such an approximation might result in either more or less than b bits being used. From (5), the set $\mathcal{D}_{ci}$ contains the MSEs resulting from bit allocations in $\mathcal{Q}_{ci}$.

The next corollary follows immediately from a lemma not given here due to limited space. It states that the nearest (in a Euclidean distance sense) a closest integer bit allocation vector can be to the optimal real-valued bit allocation vector must occur when the bit budget is at most the number of sources.

Corollary 3.1. *For any k scalar sources,*

$$\inf_{\substack{W \in \Delta \\ b \geq 1}} \|W\| = \min_{\substack{W \in \Delta \\ 1 \leq b \leq k}} \|W\|.$$

3.1. B^{ci} Optimality

The following theorem establishes that closest integer bit allocation is actually optimal optimal integer bit allocation. That is, the theorem states that the set $\mathcal{D}_{ci}$ of closest integer bit allocation MSEs contains a single number, namely the optimal integer bit allocation MSE d^{oi}.

Theorem 3.2. *For any k scalar sources and for each $b \geq 1$,*

$$\mathcal{D}_{ci} = \{d^{oi}\}$$
$$\mathcal{Q}_{ci} = \mathcal{Q}_{oi}.$$

Proof sketch. For any $B \in \mathcal{F}_I(b)$ and for any $\hat{B} \in \mathcal{Q}_{ci}$, let d and $\hat{d}$ denote the resulting MSEs, respectively. We prove that $d \geq \hat{d}$, with equality if and only if $B \in \mathcal{Q}_{ci}$.

First consider the inequality. Let $B \in \mathcal{F}_I(b)$, $\hat{B} \in \mathcal{Q}_{ci}$, and define

$$\beta^+ = \{i : B_i - \hat{B}_i > 0\}$$
$$\beta^- = \{i : B_i - \hat{B}_i < 0\}.$$

Consider any sequence of integral bit allocation vectors $\hat{B} = B^{(0)}, \ldots, B^{(n)} = B$ such that for each $m = 0, \ldots, n-1$ there exists $i \in \beta^+$ and $j \in \beta^-$ such that

$$B_l^{(m+1)} - B_l^{(m)} = \begin{cases} 1 & \text{if } l = i \\ -1 & \text{if } l = j \\ 0 & \text{otherwise}. \end{cases} \tag{8}$$

Such a sequence is guaranteed to exist since $|\hat{B}| = |B|$. For each m, let $d^{(m)}$ be the MSE achieved by $B^{(m)}$. It suffices to show that $d^{(m)}$ is monotonic nondecreasing in m.

Let $\beta = \hat{B} - B^{or}$. The construction of the sequence $B^{(m)}$ implies that $(B^{(m)} - \hat{B})_i \geq 0$ and $(B^{(m)} - \hat{B})_j \leq 0$, and therefore by a lemma not given here due to limited space,

$$-\beta_i - (B^{(m)} - \hat{B})_i \leq -\beta_j - (B^{(m)} - \hat{B})_j + 1$$

or equivalently (by the definition of β)

$$(B^{or} - B^{(m)})_i - 1 \leq (B^{or} - B^{(m)})_j \tag{9}$$

where $i \in \beta^+$ and $j \in \beta^-$ are defined by (8) (and are functions of m). Then (9) and (5) give

$$d^{(m)} = hg4^{-b/k} \cdot \sum_{l=1}^{k} 4^{(B^{or} - B^{(m)})_l} \leq hg4^{-b/k} \cdot \sum_{l=1}^{k} 4^{(B^{or} - B^{(m+1)})_l} = d^{(m+1)}$$

and therefore $\mathcal{Q}_{ci} \subset \mathcal{Q}_{oi}$.

In the other direction, let $B \in \mathcal{Q}_{oi}$. Since all optimal bit allocation vectors yield the same distortion, from the first part of the proof, it must be the case that all closest integer bit allocations give the same distortion as B. That is, $d^{(n)} = \hat{d}$, and therefore $B^{(n)} \in \mathcal{Q}_{ci}$. If $d^{(n)} = \hat{d}$, then reversing the steps of the above argument implies that for each $m = 0, \ldots, n-1$

$$-\beta_i - (B^{(m)} - \hat{B})_i = -\beta_j - (B^{(m)} - \hat{B})_j + 1. \tag{10}$$

Since $(B^{(m)} - \hat{B})_i \geq 0$ and $(B^{(m)} - \hat{B})_j \leq 0$, equality in (10) and a lemma not given here due to limited space imply that for each $m = 0, \ldots, n-1$

$$\begin{aligned} (B^{(m)} - \hat{B})_i &= 0 \\ (B^{(m)} - \hat{B})_j &= 0 \\ \beta_j - \beta_i &= 1. \end{aligned} \tag{11}$$

Recall that $i \in \beta^+$ and $j \in \beta^-$ are defined by (8), and are functions of m. If $m = 0$, then (11), a lemma not given here due to limited space, and the definition of $B^{(m)}$ imply $B^{(1)} \in \mathcal{Q}_{ci}$. Also, if $m = l$ and $B^{(l)} \in \mathcal{Q}_{ci}$, then (11), a lemma not given here due to limited space, and the definition of $B^{(m)}$ imply $B^{(l+1)} \in \mathcal{Q}_{ci}$. Hence, by induction, $B^{(n)} \in \mathcal{Q}_{ci}$. ■

4. Distortion Penalty for Optimal Integer Bit Allocations

For any k scalar sources and for each integer $b \geq 1$, let

$$p^{oi} = \frac{d^{oi}}{d^{or}}.$$

We call p^{oi} the *distortion penalty* resulting from optimal integer bit allocation. For any $B \in \mathcal{Q}_{oi}$, from Lemma 2.1, the definition of $\mathcal{Q}_{oi}$, and the fact that $d^{or} \leq d^{oi}$ we have

$$p^{oi} = \frac{1}{k} \sum_{i=1}^{k} 4^{(B^{or} - B)_i} \geq 1. \tag{12}$$

Theorem 4.1. *Consider k scalar sources with variances $\sigma_1^2, \ldots, \sigma_k^2$ and a bit budget b. The following three statements are equivalent:*

(i) $p^{oi} = 1$.

(ii) The optimal real-valued bit allocation is an integer bit allocation.

(iii) $\frac{1}{2}\log_2 \frac{\sigma_i^2}{g} + \frac{b \bmod k}{k} \in \mathbb{Z} \quad \forall i.$

Proof sketch. If B^{or} is integral, then $B^{or} \in \mathcal{Q}_{oi}$, so $p^{oi} = 1$, i.e. $(ii) \Longrightarrow (i)$. Conversely, suppose $p^{oi} = 1$. Then, for any $B \in \mathcal{Q}_{oi}$, by the arithmetic-geometric mean inequality, we have

$$1 = \frac{1}{k}\sum_{i=1}^{k} 4^{(B^{or}-B)_i} \geq 4^{\sum_{i=1}^{k}\frac{1}{k}(B^{or}-B)_i} = 4^{\frac{1}{k}(b-b)} = 1 \tag{13}$$

so the inequality in (13) is, in fact, equality. Thus, $B_i^{or} - B_i$ is a constant for all i, which must equal zero since $b = |B^{or}| = |B|$. This proves $(i) \Longrightarrow (ii)$. Lemma 2.1 and $(i) \Longleftrightarrow (ii)$ imply $p^{oi} = 1$ if and only if $\frac{1}{2}\log_2 \frac{\sigma_i^2}{g} + \frac{b}{k}$ is an integer for all i. The second equivalence then follows from the fact that

$$\frac{1}{2}\log_2 \frac{\sigma_i^2}{g} + \frac{b}{k} = \left(\frac{1}{2}\log_2 \frac{\sigma_i^2}{g} + \frac{b \bmod k}{k}\right) + \left(\frac{b - b \bmod k}{k}\right).$$

Thus we have $(i) \Longleftrightarrow (iii)$. ∎

Let $\|W\|$ denote the Euclidean norm of W. For any k scalar sources, define

$$\delta = \min_{\substack{B \in \mathcal{Q}_{oi} \\ b \geq 1}} \|B - B^{or}\|.$$

The quantity δ is well defined by Corollary 3.1 and Theorem 3.2.

The following theorem shows that either $B^{or} \in \mathcal{Q}_{oi}$ for all bit budgets congruent to some constant modulo k, or else B^{or} is never an element of $\mathcal{Q}_{oi}$, in which case the distortion penalty resulting from optimal integer bit allocation is bounded away from 1 for all bit budgets.

Theorem 4.2. *Consider k scalar sources.*

(i) If $\delta = 0$, then there exists a non-negative integer $n \leq k-1$ such that for each integer bit budget $b \geq 1$, the following holds:

$$p^{oi} = 1 \iff b \bmod k = n.$$

(ii) If $\delta > 0$, then for each integer bit budget $b \geq 1$, the following holds:

$$p^{oi} \geq \frac{1}{k}\left(4^{-\delta\sqrt{(k-1)/k}} + (k-1)4^{\delta\sqrt{1/(k(k-1))}}\right) > 1.$$

Proof sketch. First, suppose there exists a bit budget $\hat{b} \geq 1$ for which $B^{or} \in \mathcal{F}_I(\hat{b})$, and thus $B^{or} \in \mathcal{Q}_{oi}$ and $\delta = 0$. Therefore, for each integer $b \geq 1$, Theorem 4.1 implies that $p^{oi} = 1$ is true if and only if $B^{or} \in \mathcal{F}_I(b)$, or equivalently, using Lemma 2.1, that $b \bmod k = \hat{b} \bmod k$. Take $n = \hat{b} \bmod k$.

Next, suppose for every integer $b \geq 1$, we have $B^{or} \notin \mathcal{F}_I(b)$. Then for each $b \geq 1$ and for any $B \in \mathcal{Q}_{oi}$,

$$\|B - B^{or}\| \geq \delta > 0 \tag{14}$$

where the last inequality in (14) follows from the fact that $B^{or} \notin \mathcal{Q}_{oi}$ for all $b \geq 1$. For $u \geq 0$, define $f(u) = 4^{-u\sqrt{(k-1)/k}} + (k-1)4^{u\sqrt{1/(k(k-1))}}$. Then for each integer $b \geq 1$ and for every $B \in \mathcal{Q}_{oi}$,

$$p^{oi} \geq \frac{hg4^{-b/k}}{d^{or}} \cdot f(\|B - B^{or}\|) \tag{15}$$

$$\geq \frac{1}{k} f(\delta) \tag{16}$$

$$> 1 \tag{17}$$

where (15) follows from a lemma not given here due to limited space; (16) follows from (14) and the fact that $f(u)$ is monotone increasing for $u > 0$; and (17) follows from the arithmetic-geometric mean inequality. ∎

5. Lower Bound on the Distortion Penalty for Optimal Integer Bit Allocation

Theorem 5.1. *For each k, there exist k scalar sources, such that for any bit budget, the distortion penalty resulting from optimal integer bit allocation satisfies*

$$p^{oi} = \frac{3 \cdot 2^{(k-1)/k}}{k(4 - 4^{(k-1)/k})} > 1.$$

Proof sketch. Let $\gamma_k = \big(-k, -k+2, \ldots, k-2, k\big)/(2k+2)$ and for $1 \leq i \leq k$ and $a > 0$, let $\sigma_i^2 = a4^{(\gamma_{k-1})_i}$. Then

$$g = \left(\prod_{i=1}^{k} \sigma_i^2\right)^{1/k} = a$$

$$\frac{1}{2}\left(\log_2 \frac{\sigma_1^2}{g}, \ldots, \log_2 \frac{\sigma_k^2}{g}\right) = \gamma_{k-1}. \tag{18}$$

A lemma, not stated here due to limited space, and (18) imply that for each integer $b \geq 1$ and for every $B \in \mathcal{Q}_{oi}$, the vector $B - B^{or}$ is a component-wise permutation of γ_{k-1}.

Hence, for each integer $b > 0$,

$$\begin{aligned} p^{oi} &= \frac{1}{k}\sum_{i=1}^{k} 4^{-(\gamma_{k-1})_i} && (19) \\ &= \frac{1}{k}\sum_{i=0}^{k-1} 4^{-[(-(k-1)+2i)/2k]} \\ &= \frac{3\cdot 2^{(k-1)/k}}{k(4-4^{(k-1)/k})} \end{aligned}$$

where (19) follows from (12). Applying the arithmetic-geometric mean inequality to (19) gives $p^{oi} > 1$. ■

6. Upper Bound on the Distortion Penalty for Optimal Integer Bill Allocation

Theorem 3.2 and an unstated lemma imply that each component of any optimal integer bit allocation B differs from the corresponding component of the optimal real-valued bit allocation by less than 1. Hence, one easily obtains the bound

$$p^{oi} = \frac{hg4^{-b/k}\sum_{i=1}^{k} 4^{(B^{or}-B)_i}}{khg4^{-b/k}} < 4.$$

In the following theorem we give a tighter upper bound on the distortion penalty resulting from optimal integer bit allocation. The bound does not depend on the source distribution or the bit budget.

Theorem 6.1. *For each $k \geq 1$, for any k scalar sources, and for any bit budget, the distortion penalty resulting from optimal integer bit allocation satisfies*

$$p^{oi} \leq 4^{\tau}\left(1 - \frac{3\tau}{4}\right)$$

where $\tau = \frac{1}{k}\left\lceil \frac{4k}{3} - \frac{1}{1-4^{-1/k}} \right\rceil$.

As $k \to \infty$, the upper bound in Theorem 6.1 can be computed to become

$$p^{oi} \leq \frac{3}{e2^{1/3}\ln 2} + o(1) \approx 1.26.$$

Proof sketch. We show that

$$\sup_{b\geq 1}\ \sup_{\sigma_1^2,\ldots,\sigma_k^2} \frac{d^{oi}}{d^{or}} = 4^{\tau}\left(1 - \frac{3\tau}{4}\right)$$

where the suprema are taken over all possible k-tuples of sources with finite positive variances $\sigma_1^2, \ldots, \sigma_k^2$ and over all positive integer bit budgets b.

Define a mapping $f : \mathbb{R}^k \to \mathbb{R}$ by $f(U) = \sum_{i=1}^{k} 4^{-U_i}$. Then we have

$$\sup_{b\geq 1} \sup_{\sigma_1^2,\ldots,\sigma_k^2} \frac{d^{oi}}{d^{or}} = \frac{1}{k} \sup_{b\geq 1} \sup_{\sigma_1^2,\ldots,\sigma_k^2} \sum_{i=1}^{k} 4^{(B^{or}-B)_i} \qquad \forall B \in \mathcal{Q}_{ci} \tag{20}$$

$$= \frac{1}{k} \sup_{U\in V_{k-1}(0)} f(U) \tag{21}$$

$$= \frac{1}{k} \max_{0\leq j\leq k-1} \sum_{i=1}^{k} 4^{-c(k-1,j)_i} \tag{22}$$

$$= \max_{0\leq j\leq k-1} 4^{j/k} \left(1 - \frac{3}{4k} j\right) \tag{23}$$

where (20) follows from (12) and Theorem 3.2; (21) follows from an unstated lemma (due to limited space); (22) follows from the fact that the restriction of the convex function f to the closed and bounded polytope $V_{k-1}(0)$ achieves a global maximum (e.g., see [5, Theorem 6.12 on p. 154]), on the polytope's set of vertices, [1, pp. 461–462] which consists of all coordinate permutations of $c(k-1,0), \ldots, c(k-1,k-1)$; and (23) follows from (7).

For $j = 0, \ldots, k-1$, define

$$g(j) = 4^{j/k} \left(1 - \frac{3}{4k} j\right).$$

Since $g(j) > 0$ if and only if $j < 4k/3$, the function g must attain it's maximum when $j < 4k/3$. In the range $0 \leq j < 4k/3$, the ratio

$$\frac{g(j+1)}{g(j)} = 4^{1/k} \left(1 - \frac{1}{\frac{4k}{3} - j}\right)$$

is greater than 1 if and only if

$$j < \frac{4k}{3} - \frac{1}{1 - 4^{-1/k}}$$

so g attains its maximum when $j = \lceil \frac{4k}{3} - \frac{1}{1-4^{-1/k}} \rceil$. ■

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