Optimality of the Natural Binary Code for Quantizers with Channel Optimized Decoders[1]

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I. Introduction

One approach to improving the performance of scalar quantization in the presence of channel noise is to add explicit error control coding. An alternative low-complexity approach is to add to the quantizer an index assignment, which permutes the binary words associated with each encoding cell prior to transmission over the channel, and then unpermutes the binary words at the receiver prior to assigning a reproduction point at the output. The benefit of an index assignment is derived from the fact that reproduction codepoints that are relatively close on the real line can be assigned binary words which are close in the Hamming sense on average. Thus when channel errors occur, the mean squared error impact on the quantizer is reduced.

While index assignments can improve the robustness of quantizers designed for noiseless channels to the degradation caused by channel noise, another low-complexity approach is to use quantizers whose encoders and/or decoders are designed for the channel's statistical behavior. It is known that an optimal quantizer for a noiseless channel must satisfy what are known as "nearest neighbor" and "centroid" conditions on its encoder and decoder, respectively. For discrete memoryless channels it is known that an optimal quantizer must satisfy what we call "weighted nearest neighbor" and "weighted centroid" conditions on its encoder and decoder, respectively [2].

To better understand optimal quantization for noisy channels, the performance of scalar quantizers with uniform encoders and channel optimized decoders (i.e. that satisfy the weighted centroid condition) are examined for uniform sources, binary symmetric channels, and certain previously studied index assignments. The mean squared errors (MSE) for the Natural Binary Code and for randomly chosen index assignments are calculated, and the Natural Binary Code is shown to be mean squared optimal among all possible index assignments. In contrast, it is shown that asymptotically an arbitrarily large fraction of index assignments achieve an MSE arbitrarily close to the worst possible MSE.

II. Decoder Optimized Uniform Quantizers

Let a *decoder optimized uniform quantizer* denote a rate n quantizer with a uniform encoder on $[0, 1]$ and a channel optimized decoder (i.e. satisfying the weighted centroid condition), along with a uniform source on $[0, 1]$, and a binary symmetric channel with bit error probability ϵ. Let a *channel unoptimized uniform quantizer* denote a rate n uniform quantizer on $[0, 1]$, along with a uniform source on $[0, 1]$, and a binary symmetric channel with bit error probability ϵ.

Theorem 1 *The mean squared error of a decoder optimized uniform quantizer is at most the source variance, and for n sufficiently large, an arbitrarily large fraction of index assignments achieve a mean squared error arbitrarily close to the source variance.*

Since most index assignments are asymptotically bad, their average is bad as well. More precisely, the next theorem shows that the average asymptotic MSE of a decoder optimized uniform quantizer with a randomly chosen index assignment converges to the source variance, consistent with Theorem 1. Let $D_{DO}^{(RAN)}$ be a random variable denoting the MSE of a decoder optimized uniform quantizer with a randomly chosen index assignment.

Theorem 2 *The average mean squared error of a decoder optimized uniform quantizer with an index assignment chosen uniformly at random is*

$$E[D_{DO}^{(RAN)}] = \frac{2^{-2n}}{12} + \frac{1}{12} + \frac{1-(2^n+1)(1-2\epsilon+2\epsilon^2)^n}{12 \cdot 2^n}.$$

Although Theorems 1 and 2 indicate that asymptotically most index assignments yield mean squared errors close to the source variance, the next two theorems show that asymptotically, the Natural Binary Code performs better.

Theorem 3 *The mean squared error of a decoder optimized uniform quantizer with the Natural Binary Code index assignment is*

$$D_{DO}^{(NBC)} = \frac{2^{-2n}}{12} + \frac{\epsilon(1-\epsilon)}{3}\left(1-2^{-2n}\right).$$

Crimmins et al. [1] and McLaughlin, Neuhoff, and Ashley [3] showed that the Natural Binary Code is optimal for a channel unoptimized uniform quantizer. The next theorem extends this result to decoder optimized uniform quantizers.

Theorem 4 *The Natural Binary Code index assignment is optimal for a decoder optimized uniform quantizer.*

References

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