Nonreversibility of Multiple Unicast Networks *

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Abstract

We prove that for any finite directed acyclic network, there exists a corresponding multiple unicast network, such that for every alphabet, each network is solvable if and only if the other is solvable, and, for every finite field alphabet, each network is linearly solvable if and only if the other is linearly solvable. The proof is constructive and creates an extension of the original network by adding exactly $s + 5m(r-1)$ new nodes where, in the original network, m is the number of messages, r is the average number of receiver nodes demanding each source message, and s is the number of messages emitted by more than one source. The construction is then used to create a solvable multiple unicast network which becomes unsolvable over every alphabet size if all of its edge directions are reversed and if the roles of source-receiver pairs are reversed.

1. Introduction

A *network* here will refer to a finite, directed, acyclic multigraph, some of whose nodes are information sources or receivers (e.g. see [7]). Associated with the sources are *messages*, which are assumed to be arbitrary elements of a fixed finite alphabet of size at least 2. At any node in the network, each out-edge carries an alphabet symbol which is a function (called an *edge function*) of the symbols carried on the in-edges to the node, and/or a function of the node's message symbols if it is a source. Associated with each receiver are *demands*, which are a subset of all the messages of all the sources. Each receiver has *decoding functions* which map the receiver's inputs to symbols in an attempt to produce the messages demanded at the receiver. The goal is for each receiver to deduce its demanded messages from its in-edges and sources by having information propagate from the sources through the network. Each edge is allowed to be used at most once (i.e., at most 1 symbol can travel across each edge). Throughout this paper, if a network node in a figure is labeled by say x (inside a circle), then we refer to the node as n_x and we refer to an edge connecting n_x and n_y as $e_{x,y}$.

A network *code* is a collection of edge functions, one for each edge in the network, and decoding functions, one for each demand of each node in the network. A network *solution* is a network code which results in every receiver being able to compute its demands via its demand functions. A network is said to be *solvable* if it has a solution over some alphabet. A

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network is *linearly solvable* over a particular finite field alphabet if it has a solution consisting of only linear edge functions and linear decoding functions over the field. A *multiple unicast* network is a network for which every source message is emitted by exactly one source node and is demanded by exactly one receiver node. Multiple unicast networks thus consist of communications between collections of pairs of nodes.

The solvability and linear solvability of networks have recently been a subject of interest. For example, it was shown in [5] that solvable multicast networks are always linearly solvable. The class of multiple unicast networks has also been studied in various contexts.

We prove that for any network, there exists a corresponding multiple unicast network, such that for every alphabet, each network is solvable if and only if the other is solvable, and, for every finite field alphabet, each of the two networks is linearly solvable if and only if the other is linearly solvable (Theorem 2.1). The proof is constructive and creates an extension of the original network by adding exactly $s+5m(r-1)$ new nodes where, in the original network, m is the number of messages, r is the average number of receiver nodes demanding each source message. and s is the number of messages emitted by more than one source.

The *reverse* of a network $\mathcal{N}$ is a network $\mathcal{N}'$ satisfying the following:

1. The nodes of $\mathcal{N}'$ are the same as in $\mathcal{N}$.
2. The edges of $\mathcal{N}'$ are the same as in $\mathcal{N}$ but each in the reversed direction.
3. Each node which emits messages in $\mathcal{N}$, instead demands the same messages in $\mathcal{N}'$.
4. Each node which demands messages in $\mathcal{N}$, instead emits the same messages in $\mathcal{N}'$.

A network is said to be *reversible* if its reverse is solvable. A network is *linearly reversible* if its reverse is linearly solvable. Note that the reverse of a multiple unicast network is also multiple unicast.

Clearly, if a multiple unicast network has a routing solution, then it is reversible, by simply reversing the direction of information flow of the given routing solution. However, if network coding is used, then reversibility is not as straightforward. It was shown, however, in [3, 4, 6], that all linearly solvable multiple unicast networks are linearly reversible over the same alphabet. In [3, 4], an elegant "duality" principle is given, connecting algebraic coding theory and linearly reversible networks, and applications of reversibility are discussed. In [6], a network is given which has a binary (nonlinear) solution but whose reverse does not have a binary solution.

However, it has been an open question whether a solvable network could be nonreversible (i.e., over all alphabets). Clearly (in light of the results in [3,4,6]), to achieve such a result, one would need to use a network which never has a linear solution over any finite field alphabet.

We modify a solvable network constructed in [1] in such a way that it becomes unsolvable (Lemma 3.5). We show that the solvable network and the unsolvable network are either both reversible or both not reversible. Thus at least one of the two networks demonstrates the existence of a solvable nonreversible network. We then modify these two networks, using the construction presented in the first part of this paper, so that they become multiple unicast, while preserving the solvability properties. This proves that there exists a solvable nonreversible multiple unicast network. Finally, we reveal which of the two candidate solvable nonreversible multiple unicast networks is the true one (Theorem 3.9). Due to space limitations, all proofs are omitted from this manuscript and will be included in a future journal version of this work.

2. Modifying Networks into Multiple Unicast Networks

In this section, we give a construction which creates a new network from an arbitrary network containing at least some message demanded by more than one receiver. The new network has one fewer receivers demanding a particular message than the original network. Also, the new network is solvable if and only if the original network is solvable, and this property holds for linear solvability as well. As a result, if the construction is iteratively applied to each new network until no source messages are demanded by more than one receiver, then a multiple unicast network is achieved with the same solvability as the original network.

Definition. *Two networks $\mathcal{N}$ and $\mathcal{N}'$ are CSLS-equivalent if the following two conditions hold:*

1. *For any alphabet $\mathcal{A}$, $\mathcal{N}$ is solvable over $\mathcal{A}$ if and only if $\mathcal{N}'$ is solvable over $\mathcal{A}$.*
2. *For any finite field F and any positive integer k, $\mathcal{N}$ is vector solvable over F in dimension k if and only if $\mathcal{N}'$ is vector solvable over F in dimension k.*

Theorem 2.1. *Any network is CSLS-equivalent to a multiple unicast network.*

Suppose that in the original network s messages were emitted by two or more sources (e.g., if the network had 3 messages, which were emitted by 4 sources, 1 source, and 5 sources, respectively, then $s = 2$). Then s new nodes were added at the first stage of the construction. Suppose that in the original network, the ith (of m total) source message is demanded by d_i receiver nodes. Then, $d_i - 1$ iterations of the construction above will create a new network with the same solvability and where exactly one receiver node demands this message. Thus, the total number of iterations needed to avoid any messages being demanded by two or more receivers is $(d_1 - 1) + \cdots + (d_m - 1)$. Each such iteration adds 5 new nodes. If we define

$$r = \frac{1}{m} \sum_{i}^{m} d_i$$

then we can summarize this fact in the Corollary 2.2.

Corollary 2.2. *For any directed acyclic network, there exists a multiple unicast network with $s + 5m(r-1)$ additional nodes which is solvable if and only if the original network is solvable, where, in the original network, m is the number of messages, r is the average number of receiver nodes demanding each source message, and s is the number of messages emitted by more than one source. The same result holds for linear solvability as well.*

3. Nonreversibility of Multiple Unicast Networks

The classification of reversible networks is of theoretical interest and has been considered in [3, 4, 6]. In this section, we demonstrate that not all solvable multiple unicast networks are reversible.

First, note that a very simple example of a solvable network that is not reversible is shown in Figure 1. The network is trivially solvable by sending message x along the edge $e_{1,3}$, and yet the network is not reversible since there is no way to get message x from n_3 to n_2. This network is redundant in the sense that the source node n_2 could be removed while retaining the network's solvability. If n_2 is removed, then the network becomes reversible too.

One could, more specifically, consider the reversibility of *minimal* solvable networks, namely those for which no edge or source node can be removed without causing the network to become unsolvable. However, the following small example demonstrates the difficulty with such an approach.

Figure 2 gives an example of a minimal network which is linearly solvable over every alphabet size (by taking $e_{4,5} = x + y$), and yet the network is not reversible (since in the reverse network, the demand y at n_3 cannot be met).

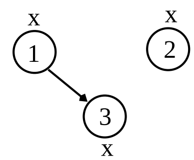

Figure 1: An example illustrating that a linearly solvable network might not be reversible if the network is not multiple unicast. Nodes n_1 and n_2 both are sources emitting the message x, and nodes n_3 is a receiver, demanding message x.

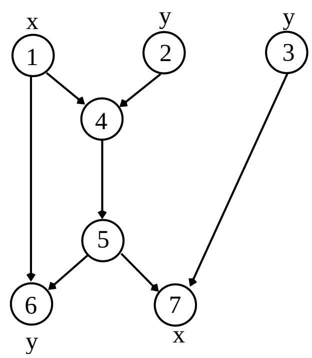

Figure 2: An example illustrating that a linearly solvable network might not have a solvable reverse if the network is not multiple unicast. Node n_1 is a source emitting message x, and nodes n_2 and n_3 are sources, both emitting message y. Nodes n_6 and n_7 are receivers, demanding messages y and x, respectively.

An alternative direction to pursue for classifying reversibility is to examine multiple unicast networks. In [3, 4, 6], the interesting result that every linearly solvable network is linearly reversible was shown. In [6], a (nonlinearly) solvable network was demonstrated which is not reversible over the binary alphabet. However, up to now, it has been an open question whether or not all solvable multiple unicast networks are reversible (i.e. over all alphabets). We prove here that not all solvable multiple unicast networks are reversible.

Our approach exploits results from [1], [2], and the first part of the present paper. More general versions of Lemma 3.1 and Lemma 3.2 were proved in [1] and [2], respectively.

Lemma 3.1. *The network $\mathcal{N}_1$ is solvable over an alphabet $\mathcal{A}$ if and only there exists a positive integer n, such that $|\mathcal{A}| = 2^n$.*

Lemma 3.2. *For any solution to network $\mathcal{N}_2$ over an alphabet $\mathcal{A}$ and for any element $0 \in \mathcal{A}$, there exist permutations $\pi_1, \dots, \pi_6$ of $\mathcal{A}$ and a mapping $+ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $(\mathcal{A}, +)$ is an Abelian group with identity element 0, $\pi_3(0) = 0$, and*

$$\begin{aligned} w &= \pi_4(\pi_1(a) + \pi_2(b)) \\ x &= \pi_5(\pi_1(a) + \pi_3(c)) \\ y &= \pi_6(\pi_2(b) + \pi_3(c)). \end{aligned}$$

We will refer to the network shown in Figure 5 as the "Insufficiency" network. It was shown in [1] that the Insufficiency network has a nonlinear scalar solution over a 4-ary alphabet but had no vector linear solution over any finite field and any vector dimension.

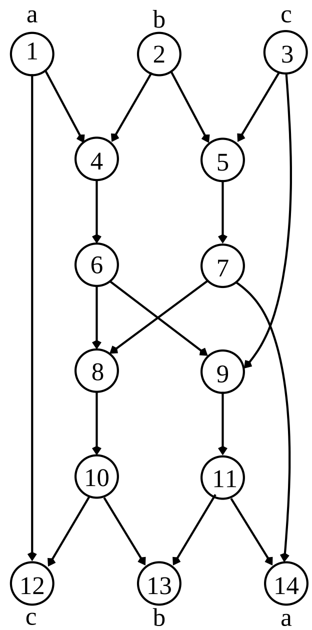

Figure 3: The network $\mathcal{N}_1$. The network is solvable only for alphabets with power-of-two cardinalities. Also, the reverse of the network is itself.

Let $\mathcal{N}_3$ be the network obtained by deleting nodes n_1, n_2, n_3 of the Insufficiency network, changing the messages at n_4, n_5, n_6 to a', b', and c', and merging nodes n_9 and n_{10}, as illustrated in Figure 6. Now create a new network $\mathcal{N}_4$, by modifying network $\mathcal{N}_3$, as shown in Figure 7. To obtain $\mathcal{N}_4$, the 6 edges in $\mathcal{N}_3$ entering receiver n_{43} are replaced by 4 new nodes (i.e., n_{x1}, n_{x2}, n_{x3}, and a new n_{43}) and 9 new edges.

Note that in the networks $\mathcal{N}_3$ and $\mathcal{N}_4$, the message c is demanded in each network by 3 receivers (n_{40}, n_{43}, and n_{46}) and all other messages are each demanded by exactly one receiver. We can create multiple unicast networks from $\mathcal{N}_3$ and $\mathcal{N}_4$ by using the technique from the first part of this paper. Specifically, create two multiple unicast networks $\mathcal{N}_5$ (see Figure 8) and $\mathcal{N}_6$ (see Figure 9), by adding gadgets to networks $\mathcal{N}_3$ and $\mathcal{N}_4$, respectively, according to the construction given in the proof of Theorem 2.1.

Lemma 3.3. *The network $\mathcal{N}_3$ is solvable.*

Corollary 3.4. *The network $\mathcal{N}_5$ is solvable.*

Lemma 3.5. *The network $\mathcal{N}_4$ is not solvable.*

Corollary 3.6. *The network $\mathcal{N}_6$ is not solvable.*

Lemma 3.7. *Network $\mathcal{N}_3$ is reversible if and only if network $\mathcal{N}_4$ is reversible.*

Lemma 3.8. *Network $\mathcal{N}_5$ is reversible if and only if network $\mathcal{N}_6$ is reversible.*

Theorem 3.9. *The reverse of the multiple unicast network $\mathcal{N}_6$ is solvable but not reversible.*

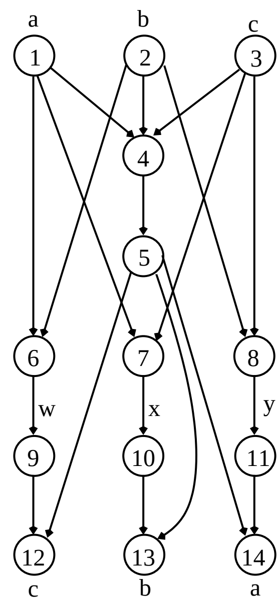

Figure 4: The network $\mathcal{N}_2$. The network's solutions are characterized in terms an Abelian group and certain fixed permutations.

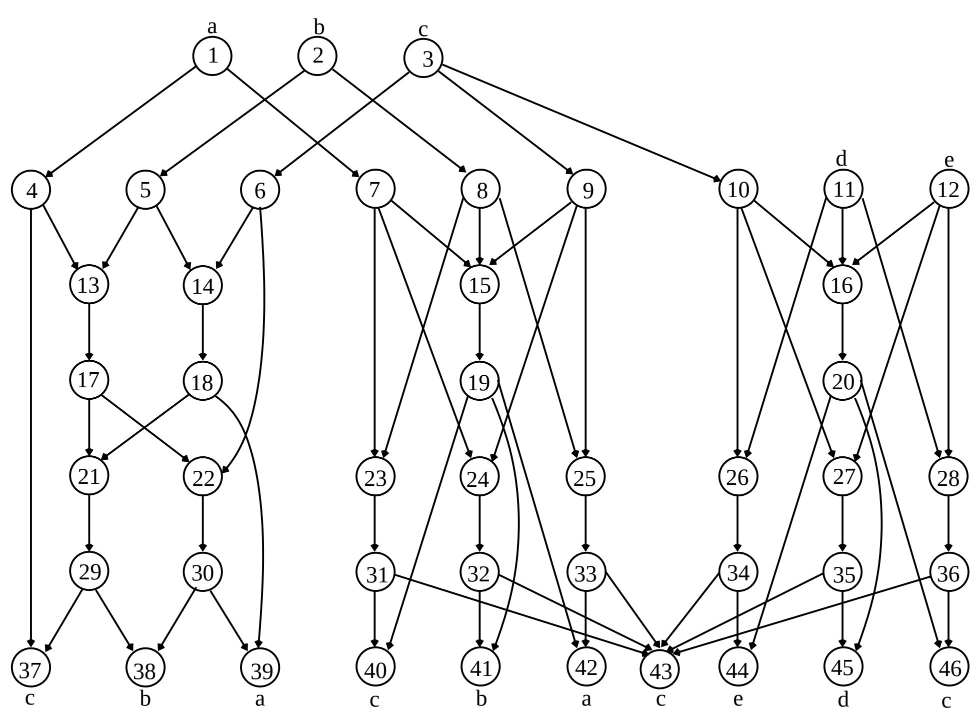

Figure 5: The Insufficiency network.

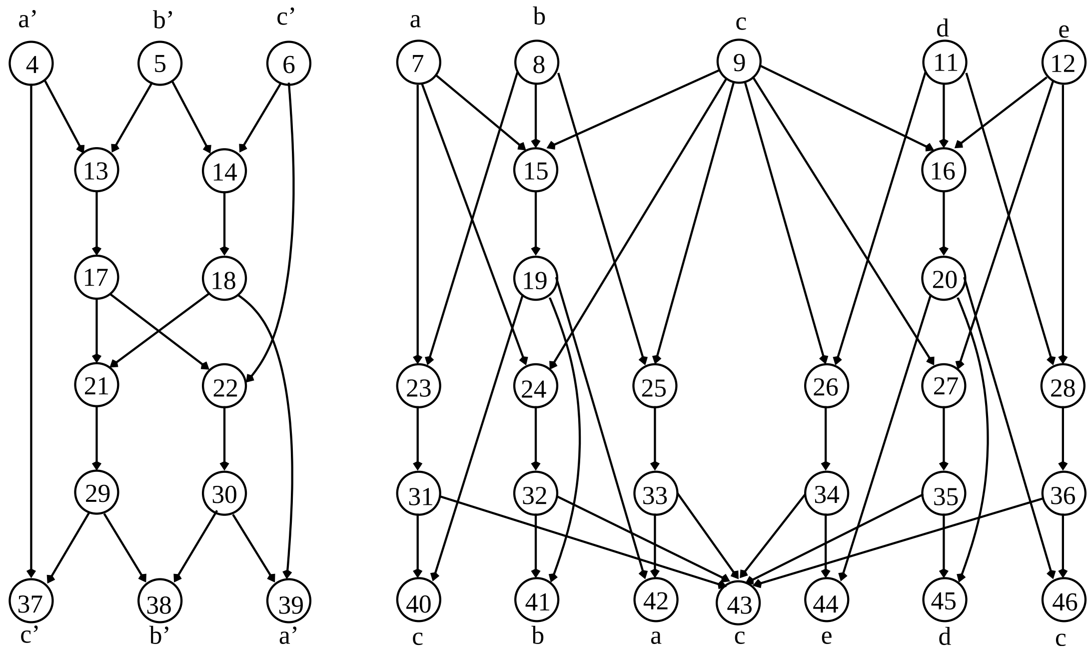

Figure 6: The network $\mathcal{N}_3$, which is a modification of the Insufficiency network.

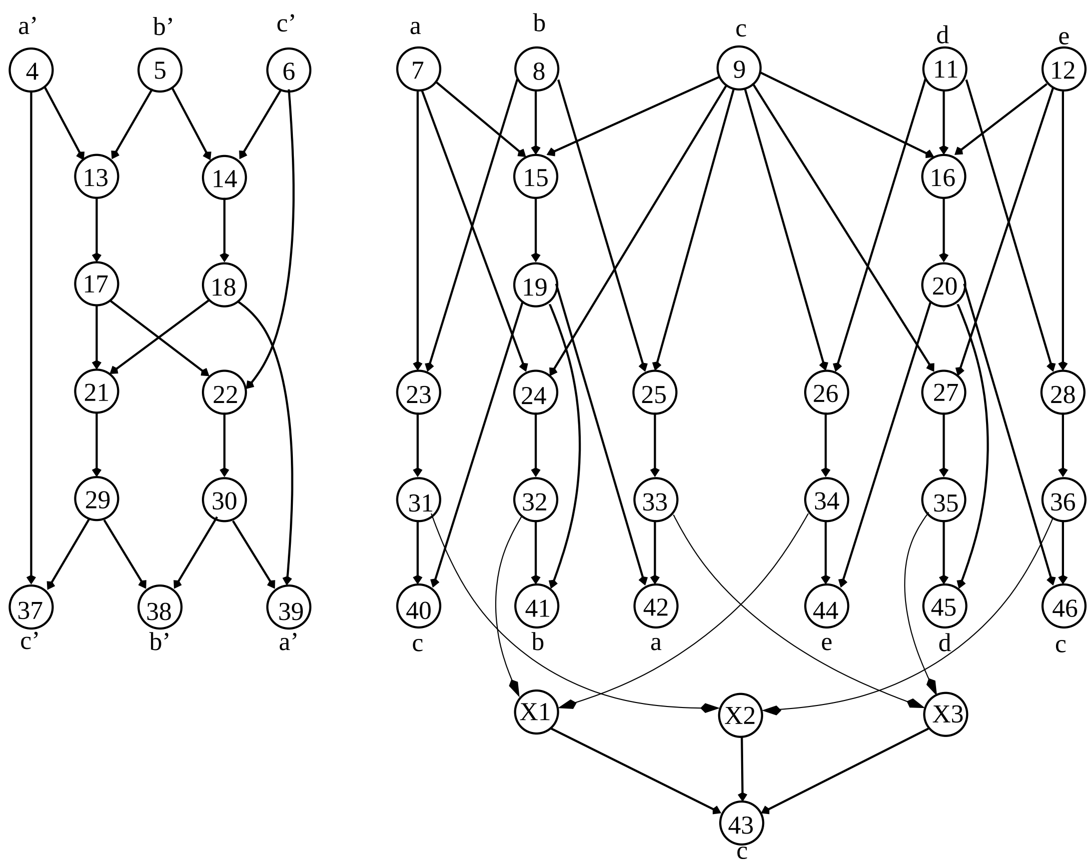

Figure 7: The network $\mathcal{N}_4$, which is a modification of network $\mathcal{N}_3$.

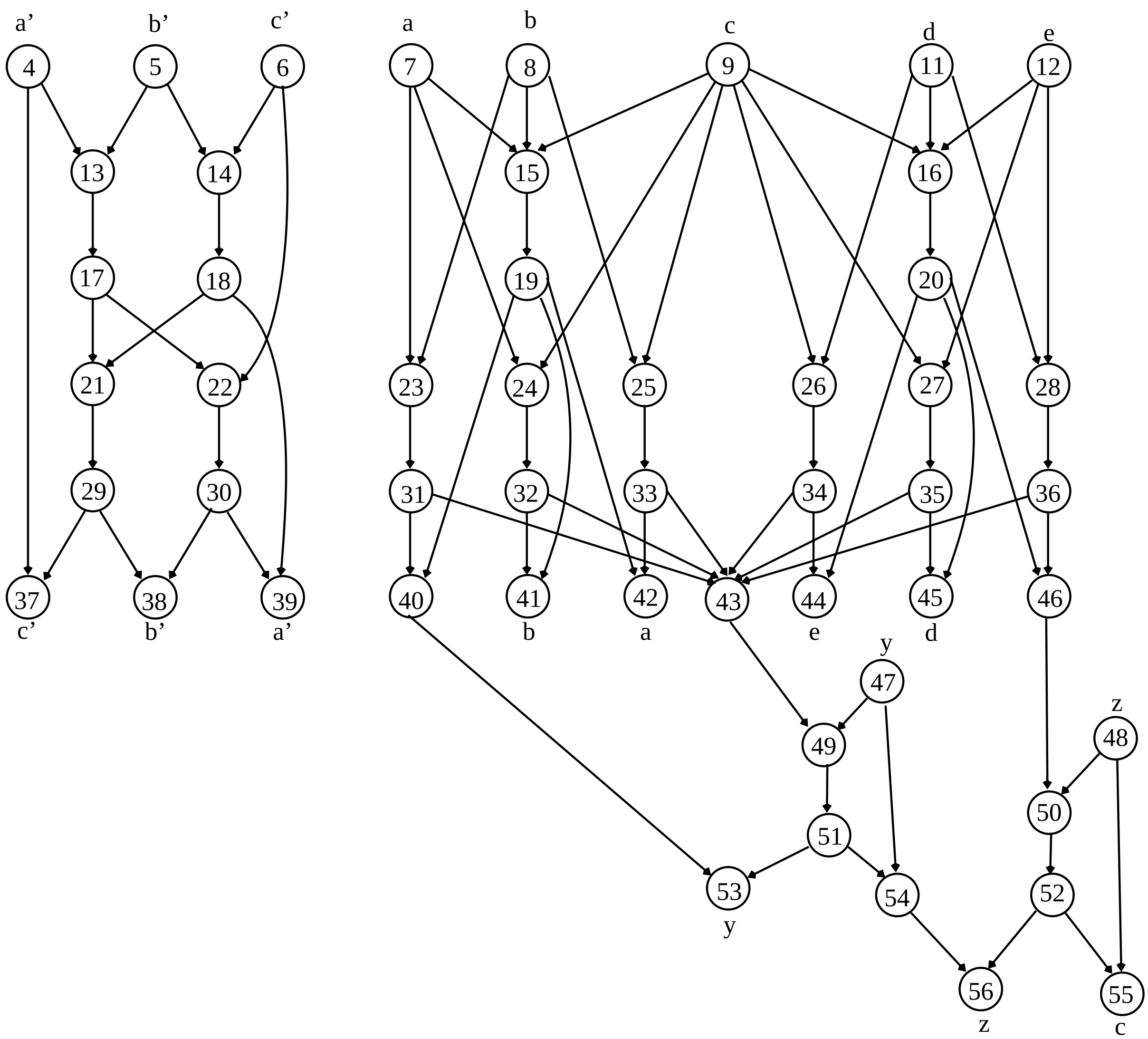

Figure 8: The multiple unicast network $\mathcal{N}_5$, which consists of $\mathcal{N}_3$ and two additional gadgets.

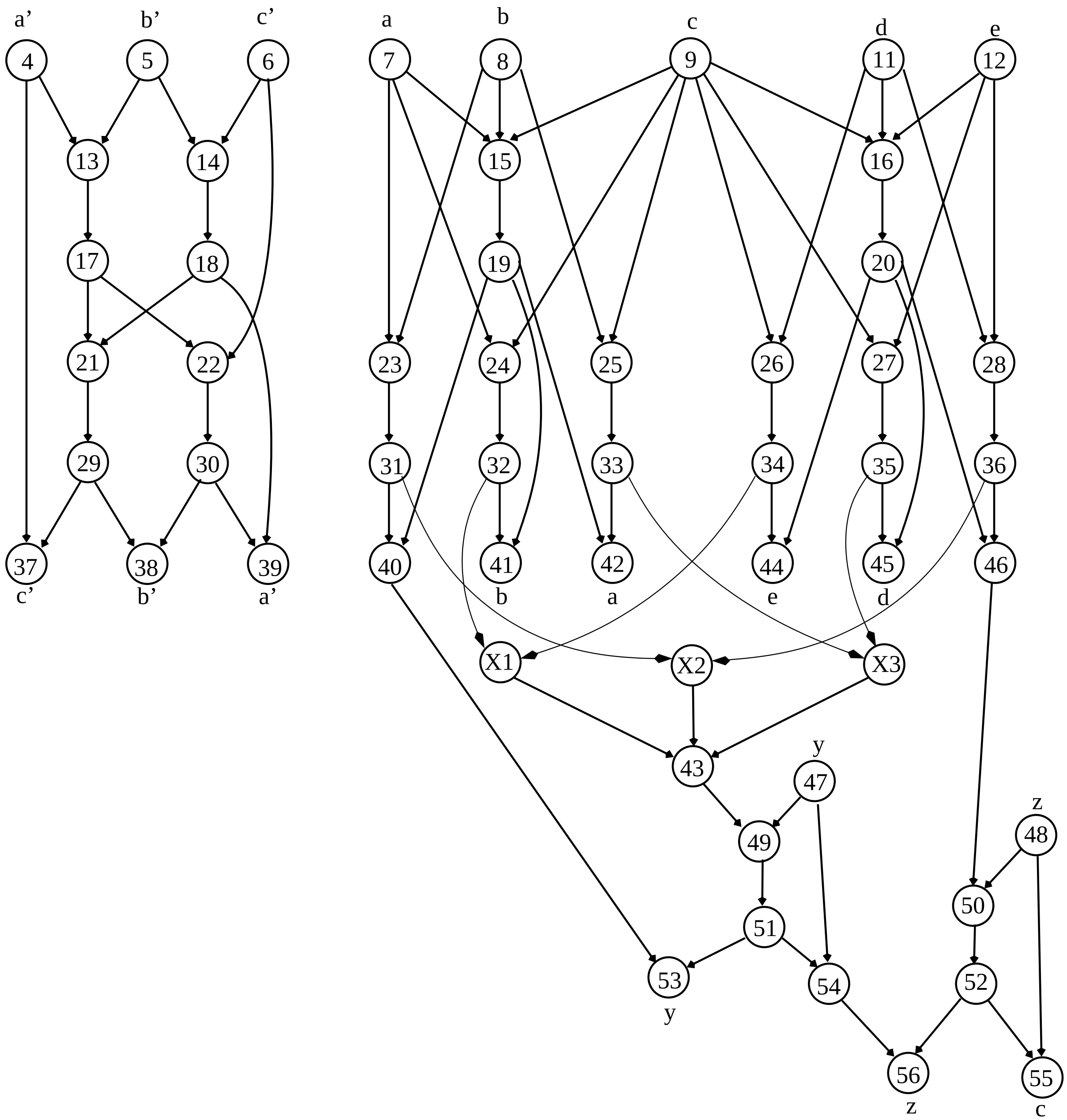

Figure 9: The multiple unicast network $\mathcal{N}_6$, which consists of $\mathcal{N}_4$ and two additional gadgets.

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