

Network Routing Capacity

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Detailed results found in:

- R. Dougherty, C. Freiling, and K. Zeger
“Linearity and Solvability in Multicast Networks”
IEEE Transactions on Information Theory
vol. 50, no. 10, pp. 2243-2256, October 2004.
- R. Dougherty, C. Freiling, and K. Zeger
“Insufficiency of Linear Coding in Network Information Flow”
IEEE Transactions on Information Theory
(submitted February 27, 2004, revised January 6, 2005).
- J. Cannons, R. Dougherty, C. Freiling, and K. Zeger
“Network Routing Capacity”
IEEE/ACM Transactions on Networking
(submitted October 16, 2004).

Manuscripts on-line at: code.ucsd.edu/zeger

Definitions

- An alphabet is a finite set.
- A network is a finite d.a.g. with source messages from a fixed alphabet and message demands at sink nodes.
- A network is degenerate if some source message cannot reach some sink demanding it.

Definitions - scalar coding

- Each edge in a network carries an alphabet symbol.
- An edge function maps in-edge symbols to an out-edge symbol.
- A decoding function maps in-edge symbols at a sink to a message.
- A solution for a given alphabet is an assignment of edge functions and decoding functions such that all sink demands are satisfied.
- A network is solvable if it has a solution for some alphabet.
- A solution is a routing solution if the output of every edge function equals a particular one of its inputs.
- A solution is a linear solution if the output of every edge function is a linear combination of its inputs (typically, finite-field alphabets are assumed).

Definitions - vector coding

- Each edge in a network carries a vector of alphabet symbols.
- An edge function maps in-edge vectors to an out-edge vector.
- A decoding function maps in-edge vectors at a sink to a message.
- A network is vector solvable if it has a solution for some alphabet and some vector dimension.
- A solution is a vector routing solution if every edge function's output components are copied from (fixed) input components.
- A vector linear solution has edge functions which are linear combinations of vectors carried on in-edges to a node, where the coefficients are matrices.
- A vector routing solution is reducible if it has at least one component of an edge function which, when removed, still yields a vector routing solution.

Definitions - (k, n) fractional coding

- Messages are vectors of dimension k .
Each edge in a network carries a vector of at most n alphabet symbols.
- A (k, n) fractional linear solution has edge functions which are linear combinations of vectors carried on in-edges to a node, where the coefficients are rectangular matrices.
- A (k, n) fractional solution is a fractional routing solution if every edge function's output components are copied from (fixed) input components.
- A (k, n) fractional routing solution is minimal if it is not reducible and if no (k, n') fractional routing solution exists for any $n' < n$.

Definitions - capacity

- The ratio k/n in a (k, n) fractional routing solution is called an achievable routing rate of the network.

- The routing capacity of a network is the quantity

$$\epsilon = \sup\{\text{all achievable routing rates}\}.$$

- Note that if a network has a routing solution, then the routing capacity of the network is at least 1.

Some prior work

- Some solvable networks do not have routing solutions (AhCaLiYe 2000).
- Every solvable multicast network has a scalar linear solution over some sufficiently large finite field alphabet (LiYeCa 2003).
- If a network has a vector routing solution, then it does not necessarily have a scalar linear solution (MéEfHoKa 2003).
- For multicast networks, solvability over a particular alphabet does not imply scalar linear solvability over the same alphabet (RaLe, MéEfHoKa, Ri 2003, DoFrZe 2004).
- For non-multicast networks, solvability does not imply vector linear solvability (DoFrZe 2004).
- For some networks, the size of the alphabet needed for a solution can be significantly reduced using fractional coding (RaLe 2004).

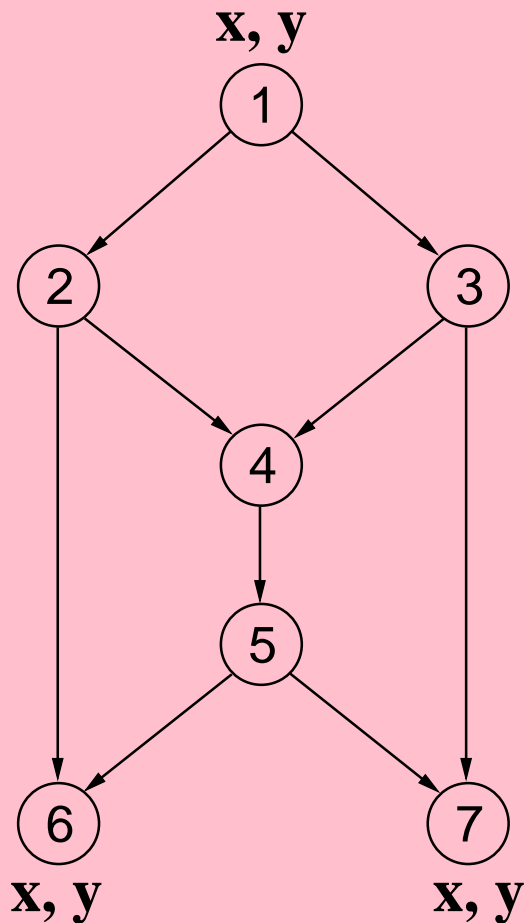
Our results

- Routing capacity definition.
- Routing capacity of example networks.
- Routing capacity is always achievable.
- Routing capacity is always rational.
- Every positive rational number is the routing capacity of some solvable network.
- An algorithm for determining the routing capacity.

Some facts

- Solvable networks may or may not have routing solutions.
- Every non-degenerate network has a (k, n) fractional routing solution for some k and n (e.g. take $k = 1$ and n equal to the number of messages in the network).

Example of routing capacity

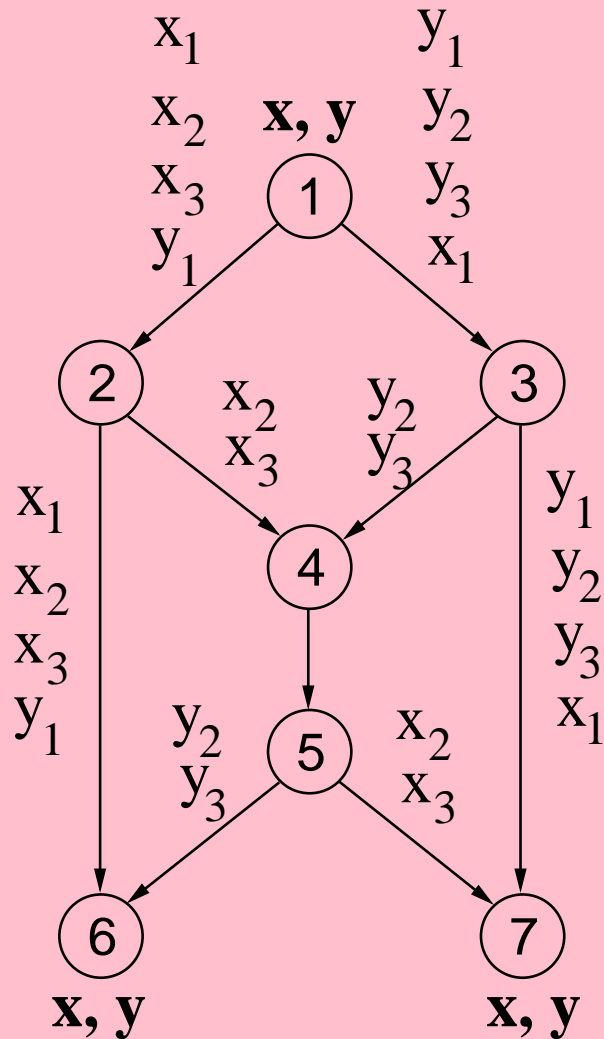


This network has a linear coding solution but no routing solution.

Each of the $2k$ message components must be carried on at least two of the edges $e_{1,2}, e_{1,3}, e_{4,5}$. Hence, $2(2k) \leq 3n$, and so $\epsilon \leq 3/4$.

Now, we will exhibit a $(3,4)$ fractional routing solution...

Example of routing capacity continued...



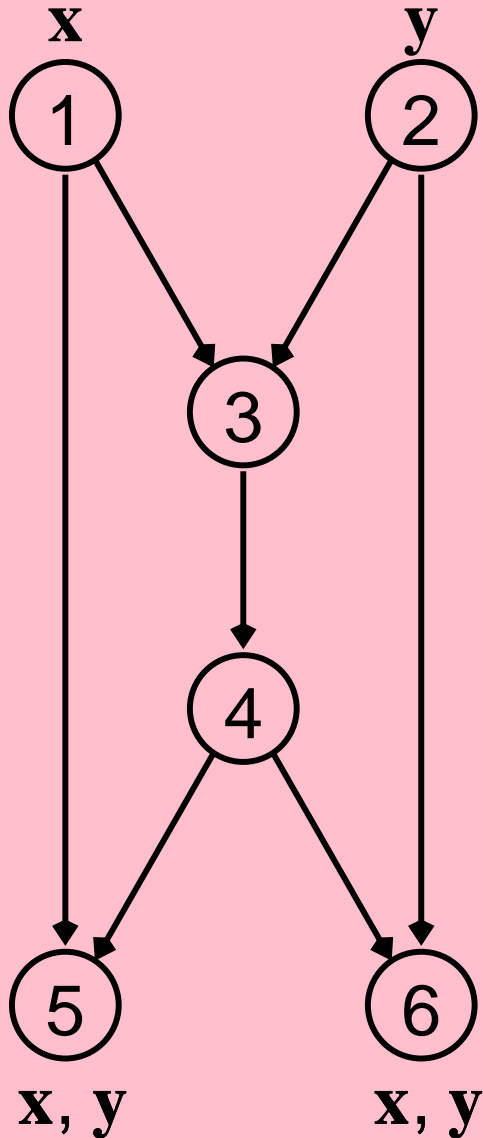
Let $k = 3$ and $n = 4$.

This is a fractional routing solution.

Thus, $3/4$ is an achievable routing rate, so $\epsilon \geq 3/4$.

Therefore, the routing capacity is $\epsilon = 3/4$.

Example of routing capacity



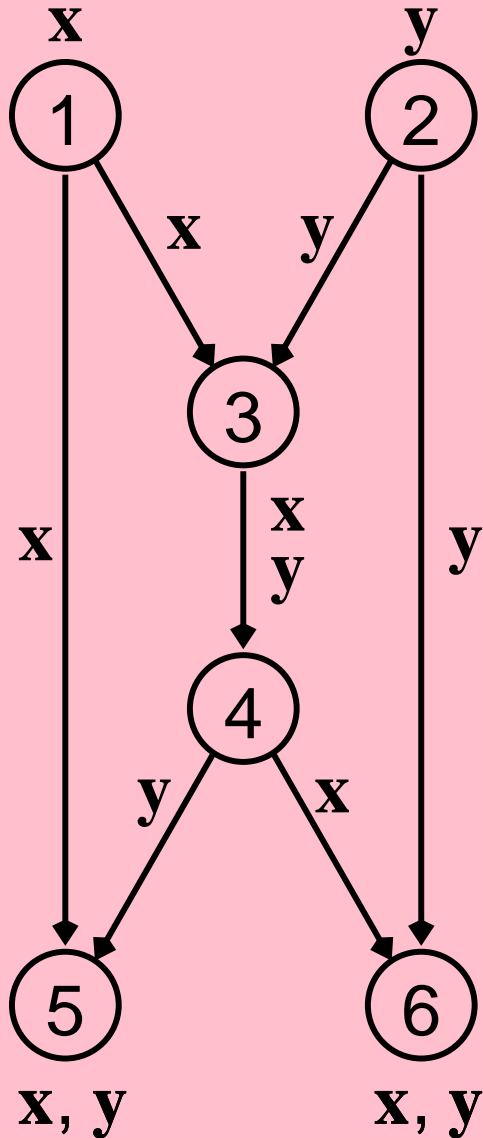
The only way to get x to n_6 is $n_1 \rightarrow n_3 \rightarrow n_4 \rightarrow n_6$.

The only way to get y to n_5 is $n_2 \rightarrow n_3 \rightarrow n_4 \rightarrow n_5$.

$e_{3,4}$ must have enough capacity for both messages.

Hence, $2k \leq n$, so $\epsilon \leq 1/2$.

Example of routing capacity continued...



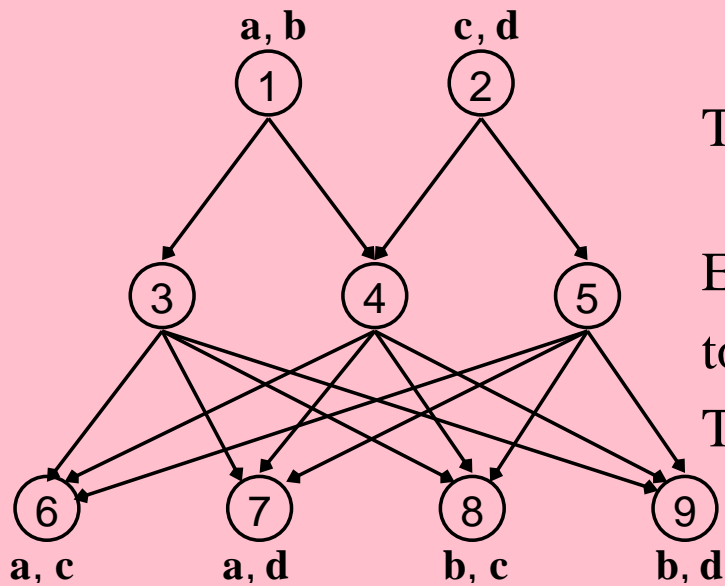
Let $k = 1$ and $n = 2$.

This is a fractional routing solution.

Thus, $1/2$ is an achievable routing rate, so $\epsilon \geq 1/2$.

Therefore, the routing capacity is $\epsilon = 1/2$.

Example of routing capacity

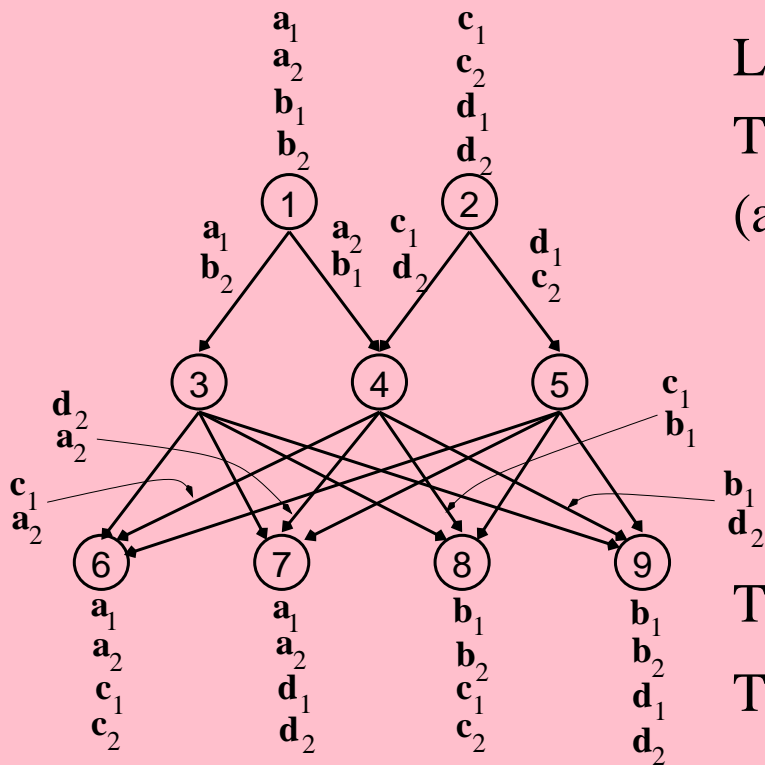


This network is due to R. Koetter.

Each source must emit at least $2k$ components and the total capacity of each source's two out-edges is $2n$.

Thus, $2k \leq 2n$, yielding $\epsilon \leq 1$.

Example of routing capacity continued...



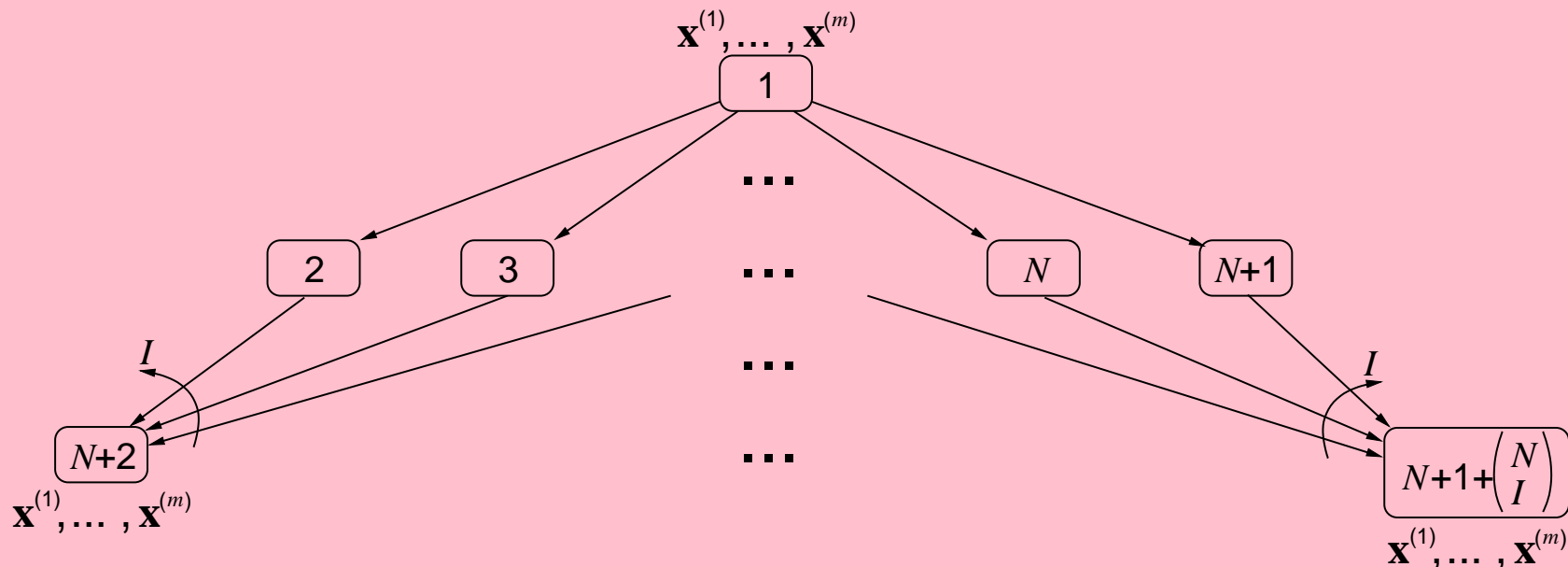
Let $k = 2$ and $n = 2$.

This is a fractional routing solution
(as given in MéEfHoKa, 2003).

Thus, $2/2$ is an achievable routing rate, so $\epsilon \geq 1$.

Therefore, the routing capacity is $\epsilon = 1$.

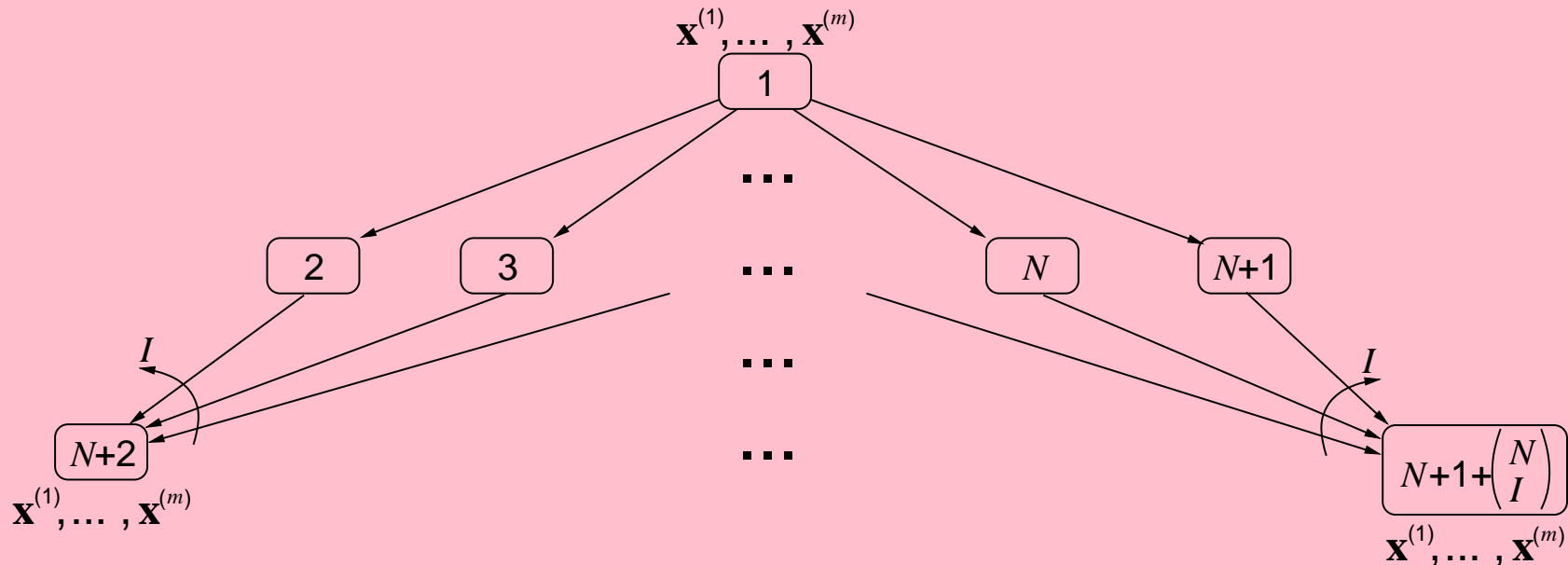
Example of routing capacity



Each node in the 3rd layer receives a unique set of I edges from the 2nd layer.

Every subset of I nodes in layer 2 must receive all mk message components from the source. Thus, each of the mk message components must appear at least $N - (I - 1)$ times on the N out-edges of the source. Since the total number of symbols on the N source out-edges is Nn , we must have $mk(N - (I - 1)) \leq Nn$ or equivalently $k/n \leq N/(m(N - I + 1))$. Hence, $\boxed{\epsilon \leq N/(m(N - I + 1))}$.

Example of routing capacity continued...



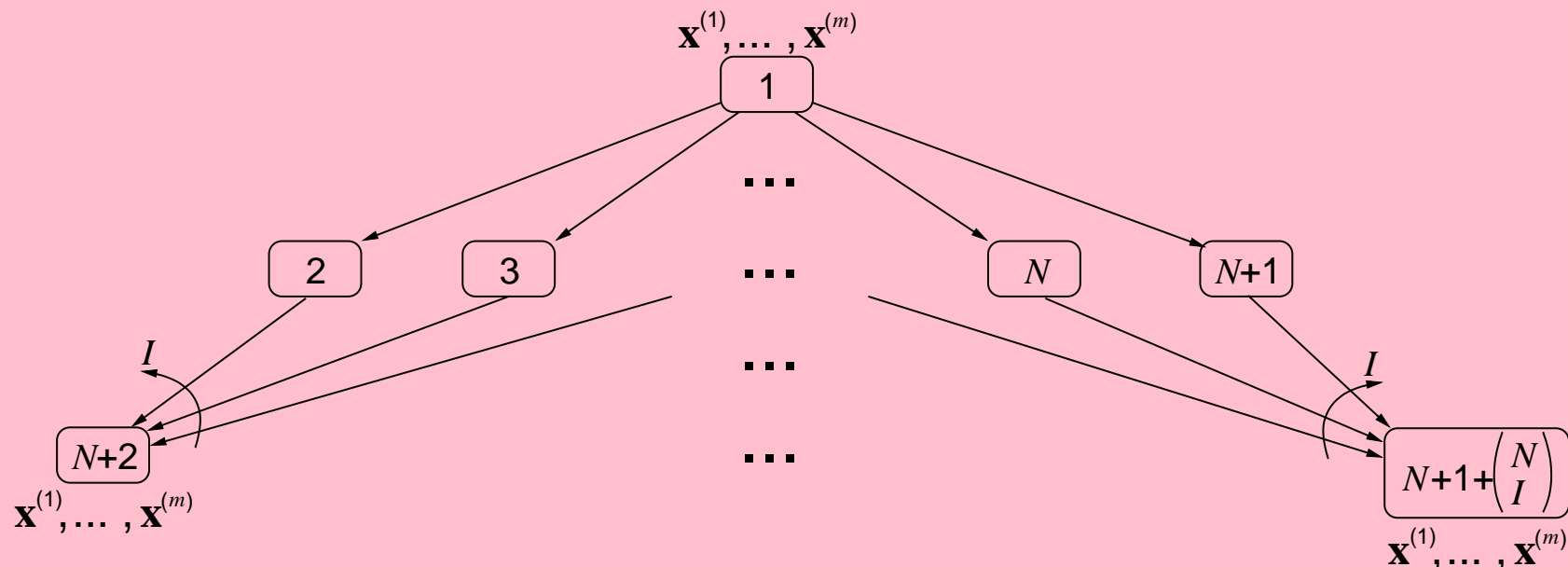
Let $k = N$ and $n = m(N - I + 1)$

There is a fractional routing solution with these parameters
(the proof is somewhat involved and will be skipped here).

Therefore, $N/(m(N - I + 1))$ is an achievable routing rate, so

$$\boxed{\epsilon \geq N/(m(N - I + 1))}$$

Therefore, the routing capacity is $\boxed{\epsilon = N/(m(N - I + 1))}$.



Some special cases of the network:

- $m = 5, N = 12, I = 8$ (AhRi 2004)

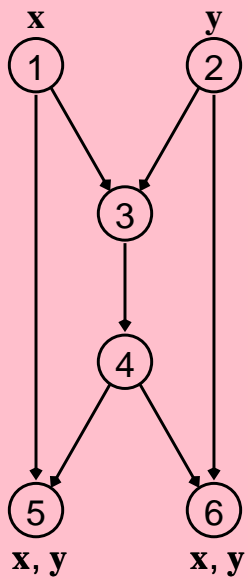
No binary scalar linear solution exist. It has a non-linear binary scalar solution using a (5, 12, 5) Nordstrom-Robinson error correcting code. We compute that the routing capacity is $\epsilon = 12/25$.

- $m = 2, N = p, I = 2$ (RaLe 2003)

The network is solvable, if the alphabet size is at least equal to the square root of the number of sinks. We compute that the routing capacity is $\epsilon = p/(2(p-1))$.

- $m = 2, N = I = 3$

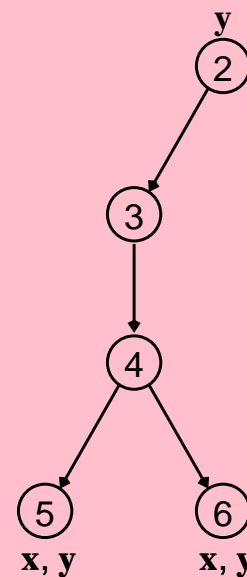
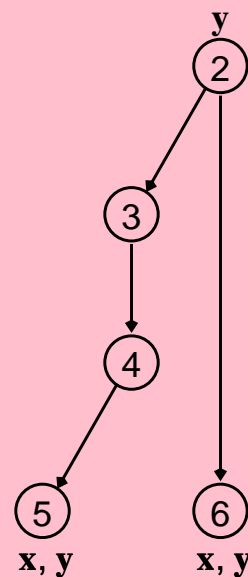
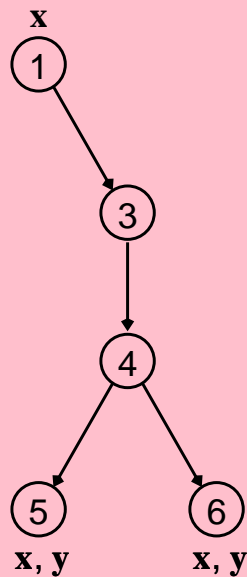
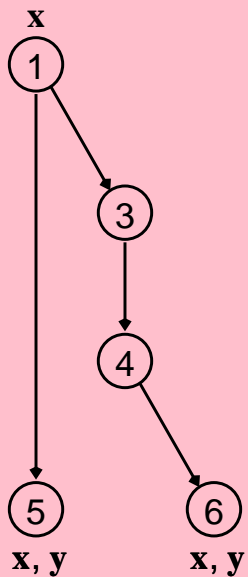
Illustrates that the network's routing capacity can be greater than 1. We obtain $\epsilon = 3/2$.



For each message m , a directed subgraph of G is an m -tree if it has exactly one directed path from the source emitting m to each destination node which demands m , and the subgraph is minimal with respect to this property (similar to directed Steiner trees).

Let T_1, T_2, \dots be all such m -trees of a network.

e.g., this network has two x -trees and two y -trees:



Define the following index sets:

$$A(\mathbf{m}) = \{i : T_i \text{ is an } \mathbf{m}\text{-tree}\}$$

$$B(e) = \{i : T_i \text{ contains edge } e\}.$$

Denote the total number of trees T_i by t .

For a given network, we call the following 4 conditions the network inequalities:

$$\sum_{i \in A(\mathbf{m})} d_i \geq 1 \quad (\forall \mathbf{m} \in M)$$

$$\sum_{i \in B(e)} d_i \leq \rho \quad (\forall e \in E)$$

$$0 \leq d_i \leq 1$$

$$0 \leq \rho \leq t$$

where d_1, \dots, d_t, ρ are real variables. If a solution (d_1, \dots, d_t, ρ) to the network inequalities has all rational components, then it is said to be a rational solution.

(kd_i represents the number of message components carried by T_i .)

Lemma: If a non-degenerate network has a minimal fractional routing solution with achievable routing rate $r > 0$, then the network inequalities have a rational solution with $\rho = 1/r$.

Lemma: If the network inequalities corresponding to a non-degenerate network have a rational solution with $\rho > 0$, then there exists a fractional routing solution with achievable routing rate $1/\rho$.

By formulating a linear programming problem, we obtain:

Theorem: The routing capacity of every non-degenerate network is achievable.

Theorem: The routing capacity of every network is rational.

Theorem: There exists an algorithm for determining the network routing capacity.

Theorem: For each rational $r > 0$ there exists a solvable network whose routing capacity is r .

Network Coding Capacity

- The coding capacity is

$$\sup \{k/n \in \mathbb{Q} : \exists(k, n) \text{ fractional coding solution}\}.$$

- routing capacity \leq linear coding capacity \leq coding capacity
- Routing capacity is independent of alphabet size.
Linear coding capacity is not independent of alphabet size.
- **Theorem:** The coding capacity of a network is independent of the alphabet used.

The End.