Sampling in 2 dimensions

Sampling refers to making the image discrete in its spatial coordinates. To discuss this, we need to introduce notation and define some functions:

The 2-D discrete delta function is defined by:

$$\delta(n_1, n_2) = \begin{cases} 1 & (n_1, n_2) = (0,0) \\ 0 & else \end{cases}$$

The 1-D continuous delta function can be defined by:

$$\delta(x) = 0 \text{ for } x \neq 0$$

and

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \delta(x) dx = 1$$

and the 2-D continuous delta function can be defined in terms of this 1-D function by:

$$\delta(x, y) = \delta(x)\delta(y)$$

in which case it is separable by definition.

The bed-of-nails function, also called an impulsive sheet, is

$$comb(x, y; \Delta x, \Delta y) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(x - j\Delta x, y - k\Delta y) = S(x, y)$$

This is composed of an infinite array of Dirac delta functions arranged in a grid of spacing $(\Delta x, \Delta y)$.

The 2-D Fourier Transform pair that we will use in this handout is:

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i2\pi(ux+vy)} dx dy$$

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{+i2\pi(ux+vy)} du dv$$

where x and y are the spatial coordinates of the original image, and u and v are the spatial frequency coordinates of the Fourier transform of the image. The extension to 3-D is obvious. This can all be written in vector notation:

$$F(\mathbf{s}) = \int_{-\infty}^{\infty} f(\mathbf{x})e^{-i2\pi\mathbf{s}^T\mathbf{x}}d\mathbf{x}$$

$$f(\mathbf{x}) = \int_{-\infty}^{\infty} F(\mathbf{s})e^{+i2\pi\mathbf{s}^T\mathbf{x}}d\mathbf{s}$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_N)$ and the units of the coordinate s_i are the inverse of the units of the corresponding spatial coordinate x_i.

Important property: if a function is separable, then its Fourier transform is separable. Using $\leftrightarrow$ to denote a F.T. pair, we have

$$
\begin{array}{rcl}
if \quad f(\mathbf{x}) & = & \prod_i f_i(x_i) \\
and \quad f(\mathbf{x}) & \leftrightarrow & F(\mathbf{s}) \\
then \quad F(\mathbf{s}) & = & \prod_i F_i(s_i), \\
where \quad f_i(x_i) & \leftrightarrow & F_i(s_i)
\end{array}
$$

Let $f_I(x, y)$ denote a continuous, infinite extent ideal image field representing the luminance, photographic density, or some desired parameter of a physical image. In a perfect sampling system, spatial samples of the ideal image would be obtained by multiplying by the spatial sampling function $S(x, y)$:

$$f_p(x, y) = f_I(x, y)S(x, y) = f_I(x, y) \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(x - j\Delta x, y - k\Delta y)$$

$$f_p(x, y) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f_I(j\Delta x, k\Delta y) \times \delta(x - j\Delta x, y - k\Delta y)$$

where it is observed that $f_I(x, y)$ may be brought inside the summation and evaluated only at the sample points $(j\Delta x, k\Delta y)$.

By the convolution theorem, the FT of the sampled image can be expressed as the convolution of the FTs of the ideal image and the sampling function:

$$\mathcal{F}_p(u, v) = \mathcal{F}_I(u, v) * \mathcal{F}_S(u, v)$$

The 2D FT of the sampling function is another infinite array of delta functions in the spatial frequency domain:

$$\mathcal{F}_S(u, v) = \frac{1}{\Delta x \Delta y} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(u - jf_{xs}, v - kf_{ys})$$

where

$$f_{xs} = \frac{1}{\Delta x} \text{ and } f_{ys} = \frac{1}{\Delta y}$$

represent the Fourier domain sampling frequencies.

If we assume that the spectrum of the ideal image is bandlimited to some bounds:

$$\mathcal{F}_I(u, v) = 0 \text{ for } |u| > f_{xc} \text{ or } |v| > f_{yc}$$

Then performing the convolution yields:

$$\mathcal{F}_P(u, v) = \frac{1}{\Delta x \Delta y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{F}_I(u - \alpha, v - \beta) \times \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(\alpha - jf_{xs}, \beta - kf_{ys}) d\alpha d\beta$$

Upon changing the order of summation and integration and invoking the sifting property of the delta function, the sampled image spectrum becomes

$$\mathcal{F}_P(u, v) = \frac{1}{\Delta x \Delta y} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \mathcal{F}_I(u - jf_{xs}, v - kf_{ys})$$

So the spectrum of the sampled image consists of the spectrum of the ideal image infinitely repeated over the frequency plane in a grid of resolution $(1/\Delta x, 1/\Delta y)$.

> *The effect of rectangular sampling in multiple dimensions is replication of the spectrum along all of the coordinate axes.*

If Δx and Δy are chosen too large with respect to the spatial frequency limits of $\mathcal{F}_I$, then the individual spectra will overlap.
The image can be reconstructed exactly from its samples if

$$\Delta x \leq \frac{1}{2f_{xc}} \Rightarrow f_{xc} \leq \frac{1}{2\Delta x} = \frac{f_{xs}}{2}$$

$$\Delta y \leq \frac{1}{2f_{yc}} \Rightarrow f_{yc} \leq \frac{1}{2\Delta y} = \frac{f_{ys}}{2}$$

$$\text{Cutoff frequency} \leq \frac{1}{2}\text{sampling frequency}$$

In physical terms the sampling period must be equal to or smaller than one-half the period of the finest spatial detail in the image. This is equivalent to the 1D sampling theorem constraint for time-varying signals. They must be sampled at a rate of at least twice the highest temporal frequency component.
If equality holds, then the sampling is at the **Nyquist rate**. If Δx and Δy are smaller than required, the image is called *oversampled*. If they are larger than required, the image is *undersampled*.

A Change of Variables Theorem

To discuss arbitrary sampling geometries, we need to establish the properties of the Fourier transform to transformations in the input coordinates of the form $\mathbf{x}' = M\mathbf{x} - \mathbf{b}$ where $\mathbf{x}', \mathbf{x}, \mathbf{b}$ are vectors and M is a matrix. By definition,

$$f(M\mathbf{x} - \mathbf{b}) \leftrightarrow \int_{-\infty}^{\infty} f(M\mathbf{x} - \mathbf{b})e^{-i2\pi\mathbf{x}^T\mathbf{s}}d\mathbf{x}$$

We will change variables:

$$\mathbf{x}' = M\mathbf{x} - \mathbf{b} \qquad \text{and} \qquad \mathbf{x} = M^{-1}(\mathbf{x}' + \mathbf{b}) \qquad \text{and} \qquad d\mathbf{x} = \frac{1}{|detM|}d\mathbf{x}'$$

We know that we can safely restrict ourselves to the cases where M is invertible because if the N-dimensional signal contains information in all directions, we will want to sample in all directions, so M must be of full rank. We obtain:

$$f(M\mathbf{x} - \mathbf{b}) \quad \leftrightarrow \quad \frac{1}{|detM|} \int_{-\infty}^{\infty} f(\mathbf{x}')e^{-i2\pi(\mathbf{x}'^T + \mathbf{b}^T)(M^{-1})^T\mathbf{s}}d\mathbf{x}' \tag{1}$$

$$= \quad \frac{e^{-i2\pi\mathbf{b}^T(M^{-1})^T\mathbf{s}}}{|detM|} \int_{-\infty}^{\infty} f(\mathbf{x}')e^{-i2\pi\mathbf{x}'^T(M^{-1})^T\mathbf{s}}d\mathbf{x}' \tag{2}$$

$$= \quad \frac{e^{-i2\pi\mathbf{b}^T(M^{-1})^T\mathbf{s}}}{|detM|} F((M^{-1})^T\mathbf{s}) \tag{3}$$

This is a compact notation containing many different special cases. As a check of correctness, let $M = I$, the identity matrix, and let $b = 0$.

Aside #1: The formula above relies on a theorem for changing variables in a double integral. For a single integral, the theorem is:

$$\int_c^x f[g(t)]g'(t)dt = \int_{g(c)}^{g(x)} f(u)du$$

In two dimensions, if we have an invertible mapping from S to T given by

$$x = X(u,v) \text{ and } y = Y(u,v)$$

then the theorem states

$$\int_S \int f(x,y)dxdy = \int_T \int f[X(u,v), Y(u,v)]|J(u,v)|dudv$$

The factor $J(u,v)$ plays the role of the $g'(t)$ which appears in the 1-D formula. $J(u,v)$ is called the *Jacobian determinant* of the mapping. It is equal to

$$J(u,v) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix}$$

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The shift theorem: Taking $M = I$ and b arbitrary, we get a multidimensional shift theorem:

$$f(\mathbf{x} - \mathbf{b}) \leftrightarrow e^{-i2\pi \mathbf{b}^T \mathbf{s}} F(\mathbf{s})$$

Two-dimensional scaling: Let

$$M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Then

$$(M^{-1})^T = \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix},$$

and

$$f(\lambda_1 x, \lambda_2 y) = f(M\mathbf{x}) \leftrightarrow \frac{1}{|\lambda_1 \lambda_2|} F((M^{-1})^T \mathbf{s}) = \frac{1}{|\lambda_1 \lambda_2|} F(\frac{1}{\lambda_1}u, \frac{1}{\lambda_2}v).$$

Rotation:
Let $MM^T = I$ where $det M = 1$. The matrix M represents a rotation. Using the change of variable theorem together with the fact that $(M^{-1})^T = (M^T)^T = M$ leads to

$$f(M\mathbf{x}) \leftrightarrow F(M\mathbf{s})$$

This means that rotating an image just rotates its Fourier Transform by the same amount.

Delta Function Scaling: Applying our change of variable expression to the delta function leads to

$$\delta(M\mathbf{x} - \mathbf{b}) \leftrightarrow \frac{e^{-i2\pi \mathbf{b}^T (M^{-1})^T \mathbf{s}}}{|det M|}$$

Applying it to the function $\delta(\mathbf{x} - M^{-1}\mathbf{b})$ leads to

$$\delta(\mathbf{x} - M^{-1}\mathbf{b}) \leftrightarrow e^{-i2\pi \mathbf{b}^T (M^{-1})^T \mathbf{s}}$$

Dividing both sides of this equation by $|det M|$, we note that the right hand side of this equation matches the right hand side of the previous one. By the uniqueness of the FT, we can conclude that the delta function scales as

$$\boxed{\delta(M\mathbf{x} - \mathbf{b}) = \frac{\delta(\mathbf{x} - M^{-1}\mathbf{b})}{|det M|}}$$

Sampling on an arbitrary lattice:

Definition: A lattice Λ_n in $\mathcal{R}^n$ is composed of all integral combinations of a set of linearly independent vectors which span the space.

In 2-D, let A be a 2×2 non-singular matrix called the generator matrix of the lattice. Then the lattice $\mathcal{L}$ is composed of

$$\mathcal{L} = \{t \in \mathcal{R}^2 / t = Am \text{ for } m \in \mathcal{Z}^2\}$$

Thus, the sampling function that would be used for sampling on an arbitrary 2-d lattice could be denoted by:

$$\sum_{m \in \mathcal{Z}^2} \delta(t - Am)$$

We define, in one dimension, a train-of-impulses function which we will call the Shah function:

$$\text{III}(x) = \sum_n \delta(x - n) \text{ where } n \in \mathcal{Z}$$

In N dimensions, this is the same as the bed-of-nails or comb function which we defined earlier, except where we take unit spacing in the N dimensions:

$$\text{III}(\mathbf{x}) = \sum_{\mathbf{n}} \delta(\mathbf{x} - \mathbf{n}) \text{ where } \mathbf{n} \in \mathcal{Z}^N$$

Now, we can use the scaling of the delta function on this to obtain:

$$\text{III}(M\mathbf{x}) = \sum_{\mathbf{n}} \delta(M\mathbf{x} - \mathbf{n}) = \sum_{\mathbf{n}} \frac{\delta(\mathbf{x} - M^{-1}\mathbf{n})}{|detM|} \tag{4}$$

The change of variables theorem, applied to the function $\text{III}(M\mathbf{x})$ with $b = 0$ yields the FT relationship:

$$f(\mathbf{x})\text{III}(M\mathbf{x}) \leftrightarrow \frac{F(\mathbf{s})}{|detM|} * \text{III}((M^{-1})^T \mathbf{s}). \tag{5}$$

Now, we can multiply both sides of this equation by $|detM|$, and use Eqn 4 to substitute for $\text{III}(M\mathbf{x})$, and similarly substitute for $\text{III}((M^{-1})^T\mathbf{s})$ to obtain

$$f(\mathbf{x}) \sum_{\mathbf{n}} \delta(\mathbf{x} - M^{-1}\mathbf{n}) \leftrightarrow \frac{F(\mathbf{s})}{|detM^{-1}|} * \sum_{\mathbf{m}} \delta(\mathbf{s} - M^T\mathbf{m}) \tag{6}$$

where we have used the fact that $detM = detM^T$. To put this equation in a pretty form, we'll denote $A = M^{-1}$ and $B = M^T$. Note that convolution with the Shah function results in replication of the spectrum. This leads to

$$\boxed{f(\mathbf{x}) \sum_{\mathbf{n}} \delta(\mathbf{x} - A\mathbf{n}) \leftrightarrow \frac{1}{|detA|} \sum_{\mathbf{m}} F(\mathbf{s} - B\mathbf{m})} \tag{7}$$

This is a sampling theorem valid for arbitrary sampling lattices. The columns of the matrix A form vectors which describe the sampling lattice. Integer combinations of these vectors describe all of the sampling locations in the function (spatial) domain. In the Fourier domain, the columns of the matrix B describe all of the replication locations of the spectrum.

If the sampling rates are high enough for a bandlimited function, the original function may be recovered. Application of an appropriate lowpass filter will provide for the recovered signal

$$\frac{F(\mathbf{s})}{|detA|}$$

$|detA|$ is the area of the parallelopiped formed by the column vectors of A.

Aside #2: Proof of the fact that $|detA|$ is the area of the parallelopiped formed by the column vectors of A. Let

$$A = \left(\begin{array}{cc} a_1 & a_2 \\ b_1 & b_2 \end{array} \right)$$

Let θ_1 denote the angle that the first vector (a_1, b_1) makes with the x-axis. Similarly, let θ_2 denote the angle that (a_2, b_2) makes with the x-axis. Call the angle between the two vectors θ_3. We use x_1, x_2 to denote the lengths of vectors 1 and 2. If we drop a perpendicular from the tip of vector 1 to vector 2, and call that perpendicular distance x_4, then we have the following relationships:

$$\sin \theta_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2}} \qquad \sin \theta_2 = \frac{b_2}{\sqrt{a_2^2 + b_2^2}}$$

$$\cos \theta_1 = \frac{a_1}{\sqrt{a_1^2 + b_1^2}} \qquad \cos \theta_2 = \frac{a_2}{\sqrt{a_2^2 + b_2^2}}$$

We can solve for x_4 by substituting the 4 expressions above into:

$$\sin \theta_3 = \frac{x_4}{\sqrt{a_1^2 + b_1^2}} = \sin(\theta_2 - \theta_1) = \sin \theta_2 \cos \theta_1 - \sin \theta_1 \cos \theta_2.$$

We use that value of x_4 in the equation

$$Area = x_4 x_2$$

together with the fact that

$$x_2^2 = a_2^2 + b_2^2$$

to obtain

$$Area = a_1 b_2 - b_1 a_2 = detA$$

For a given matrix A, the sampling density may be determined as the inverse of the area, since there is a one-to-one correspondence between lattice cells and lattice points.

Example: 2-D rectangular vs. hexagonal sampling

Consider a signal with a circularly symmetric spectrum as shown below (left). Then a rect-angular sampling pattern which does not cause aliasing would put the spectral replications as shown below (right).
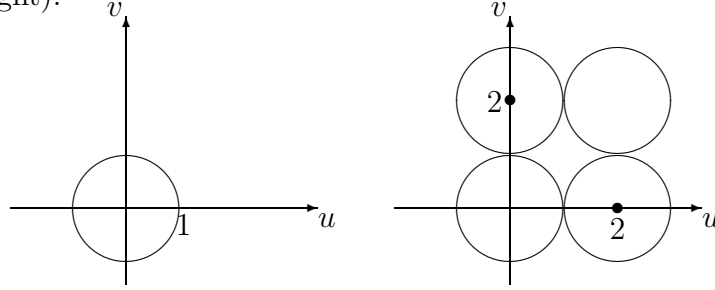
The generator matrix for the replication lattice is therefore:

$$B_r = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and so the generator matrix for the sampling matrix in the spatial domain is

$$A_r = (B_r^{-1})^T = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

and we have $det A_r = 1/4$. For the same spectrum, we could achieve no aliasing with a hexagonal lattice. The generator matrix for this is

$$B_h = \begin{pmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{pmatrix}$$

The sampling in the spatial domain occurs at lattice points given by the generator matrix

$$A_h = (B_h^{-1})^T = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix} \approx \begin{pmatrix} 0.5 & 0 \\ -0.3 & 0.6 \end{pmatrix}$$

This is a hexagonal lattice too, and it is less dense than the lattice for rectangular spatial sampling. How much less dense can be determined by the ratio of the determinants

$$\frac{\text{sampling density of hex grid}}{\text{sampling density of rect grid}} = \frac{\frac{1}{det A_h}}{\frac{1}{det A_r}} = \frac{2\sqrt{3}}{4} = 0.866$$

So hexagonal sampling is more efficient than rectangular sampling for circularly symmetric spectra. Further gains can be had for packing in 3 dimensions.