1) : The step size for a uniform quantizer with number of bits $j$ is: $s = \frac{256}{2^j}$, and the quantization levels can be written as $[\frac{s}{2} + s \cdot n]$. For example, when the number of bits is 6, values in the form of $4n$, $(4n+1)$, $(4n+2)$, $(4n+3)$ will all be mapped to $(4n+2)$. Using this quantization method, the MSE values for uniform quantizer with number of bits from 1 to 7 are:

$$MSE = 2240.6, 493.7, 101.6, 20.4, 5.1, 1.5, 0.5$$

Sanity Check: When the number of bits in the quantizer is 7, an even value $2n$ will be quantized to $2n+1$. Half of pixel values are even, so the MSE will be .5. When the number of bits is 6, pixel values in the form of $4n$, $(4n+1)$, $(4n+2)$, $(4n+3)$ will all be quantized to $(4n+2)$. With probability of one quarter for each of $4n$, $(4n+1)$, $(4n+2)$, $(4n+3)$, the expected MSE is:

$$E(error) = \frac{1}{4} \cdot (2^2 + 1 + 0 + 1) = 1.5$$

2) : Using Lloyd-max quantization method introduced in class, the quantization level and decision level will reach steady state in less than 10 iterations. In each iteration, the quantization level is the conditional expectation of the pixel values within the decision interval,

$$r_k = \frac{\sum_{u_{i,j} \in [t_k, t_{k+1})} u_{i,j}}{\text{num of } u_{i,j} \in [t_k, t_{k+1})}$$

And the decision level is $t_k = (r_k + r_{k+1})/2$

The MSE results are:

$$MSE = 679.8428 \quad 180.7731 \quad 48.9287 \quad 15.1321 \quad 4.5347 \quad 1.3115 \quad 0.4585$$

The plot of two MSE curves can be found in Fig1 in the next page.

3) : We design a piece-wise quantizer which will perform Lloyd algorithm at different regions respectively. At low grayscale value region, say $[0, 31]$, all the pixels will be mapped to one quantization level, namely the conditional expectation of this region. We then use Lloyd algorithm to quantize the rest of the pixels (pixels with value larger than 31).
A plot of quantization mapping for the piecewise quantizer is shown at the beginning of the next page.

The MSE values for a 4-bit uniform quantizer, Lloyd-Max quantizer, and piece-wise quantizer are 20.4, 15.1321, and 31.0364 respectively.

For pixel values larger than 31, compared to the piece-wise quantizer we designed, the quantization level (also the decision levels) for Lloyd-Max quantizer is sparser, which basically means that the resolution for Lloyd-Max quantizer in this region is worse than piece-wise quantizer. However, the piece-wise quantizer has a larger MSE measure. This is because piece-wise quantizer has only one quantization level to represent the background while Lloyd-Max quantizer will have more than one. Since there are a lot of background pixels, the overall distortion of piece-wise quantizer is larger than the Lloyd-Max quantizer and uniform quantizer.

4) : If we do quantization to the equalized version of the image, the MSE values for uniform quantizer are:

\[
\text{MSE} = 1394.8, \ 348.9, \ 82.6, \ 18.7, \ 5.9, \ 1.2, \ 0.5,
\]
and the MSE values for the Lloyd-Max quantizer are:

\[ \text{MSE} = 1393.5, \ 344.5, \ 80.3, \ 16.8, \ 2.9, \ 0, \ 0 \]

After global histogram equalization, the distribution of pixel values is almost uniform, so the gap between these two MSE curves is negligible when the number of quantization bits is small.

Sanity Check: When the number of quantization bits is 1. If we use continuous random variable with uniform distribution to approximate the pixel values of the equalized image. The expected value of MSE can be written as:

\[ E(error) \approx \int_{0}^{64} \frac{x^2}{64} dx \approx 1400 \]

, which is roughly the same as simulation value of MSE.

The empty-bin problem is taken care of by removing the empty bin and add extra quantization level to the quantization interval with largest MSE measure. After equalization, the pixels values are mapped to less than 64 bins, so the MSE for Lloyd-Max quantizer with 7 and 6 quantization bit is 0. The plot of two MSE curves can be found in the next page.
Fig. 3. Quantization MSE of Equalized Images