Example of Arithmetic Coding

We have 3 symbols in the alphabet, with probabilities

\[ p(1) = 0.8 \quad p(2) = 0.02 \quad p(3) = 0.18 \]

The cdf corresponding to this pdf has values:

\[ F(0) = 0 \quad F(1) = 0.8 \quad F(2) = 0.82 \quad F(3) = 1 \]

The input sequence we will code is

\[ X_1, X_2, X_3, X_4 \ldots = 1, 3, 2, 1 \ldots \]

We initialize the lower and upper endpoints of the interval to be

\[ l^{(0)} = 0 \quad u^{(0)} = 1 \]

After reading input symbol \( X_n \), we will update the sub-interval endpoints as follows:

\[ l^{(n)} = l^{(n-1)} + [u^{(n-1)} - l^{(n-1)}]F(X_n) - 1 \]
\[ u^{(n)} = l^{(n-1)} + [u^{(n-1)} - l^{(n-1)}]F(X_n) \]

We begin with the first symbol: \( X_1 = 1 \). We apply the endpoint updating equations to obtain:

\[ l^{(1)} = 0 + (1 - 0) \times 0 = 0 \]
\[ u^{(1)} = 0 + (1 - 0) \times 0.8 = 0.8 \]

This interval straddles 0.5, so there is no output of bits. So we proceed to look at the next symbol: \( X_2 = 3 \). We find the new sub-interval by applying the endpoint updating equations:

\[ l^{(2)} = 0 + (0.8 - 0)F(2) = 0.8 \times 0.82 = 0.656 \]
\[ u^{(2)} = 0 + (0.8 - 0)F(3) = 0.8 \times 1 = 0.8 \]

Now, this is entirely in the upper half of the unit interval. So we output the bit 1. Now we can rescale with the rescaling function \( E_2 \). Recall that the two possible re-scaling functions are

\[ E_1(x) = 2x \]
\[ E_2(x) = 2(x - 0.5) \]

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where $E_1$ is used for rescaling the lower half, and $E_2$ is used for rescaling the upper half. Rescaling our little interval with $E_2$:

$$l^{(2)} = 2 \times (0.656 - 0.5) = 0.312$$

$$u^{(2)} = 2 \times (0.8 - 0.5) = 0.6$$

This straddles 0.5, so there is no further release of bits at this time. We look at the next input symbol, $X_3 = 2$.

$$l^{(3)} = 0.312 + (0.6 - 0.312)F(1) = 0.5424$$

$$u^{(3)} = 0.312 + (0.6 - 0.312)F(2) = 0.54816$$

We got a very low-probability input, so the interval suddenly got very small. So now there’s going to be a release of many bits. After each one, we rescale. Currently, we’re in the upper half of the unit interval, so we release the bit 1, and rescale with $E_2$ to get:

$$l^{(3)} = 0.0848$$

$$u^{(3)} = 0.09632$$

Now it’s in the lower half interval, so release the bit 0, and rescale with $E_1$ to obtain:

$$l^{(3)} = 2 \times 0.0848 = 0.1696$$

$$u^{(3)} = 2 \times 0.09632 = 0.19264$$

Still in the lower half interval, so release another bit 0, and rescale again with $E_1$, etc.