Quantitative Example of Energy Compaction and Decorrelation

Let $u$ be a $2 \times 1$ vector with mean zero:

$$u = \begin{bmatrix} u(0) \\ u(1) \end{bmatrix}$$

and covariance

$$R_u = \begin{pmatrix} E[(u(0) - \mu_0)(u(0) - \mu_0)] & E[(u(0) - \mu_0)(u(1) - \mu_1)] \\ E[(u(0) - \mu_0)(u(1) - \mu_1)] & E[(u(1) - \mu_1)(u(1) - \mu_1)] \end{pmatrix}$$

$$= \begin{pmatrix} E[u(0)^2] & E[u(0)u(1)] \\ E[u(1)u(0)] & E[u(1)^2] \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

for $0 < \rho < 1$. From the expression for $R_u$, the variances

$$\sigma_{u(0)}^2 = \sigma_{u(1)}^2 = 1$$

that is, the total average energy of 2 is distributed equally between $u(0)$ and $u(1)$.

The parameter $\rho$ gives an indication of the correlation between $u(0)$ and $u(1)$. The correlation is by definition

$$corr[u(0), u(1)] = \frac{E[u(0)u(1)]}{\sigma_{u(0)}\sigma_{u(1)}} = \rho$$

So in this case, the off-diagonal elements of the covariance matrix are in fact exactly equal to the correlation, but that is only because the $\sigma$’s are both equal to one.

Now we transform $u$ as follows:

$$v = \frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix} u$$

The covariance matrix of $v$ is then

$$R_v = \begin{pmatrix} 1 + \sqrt{3}\rho/2 & \rho/2 \\ \rho/2 & 1 - \sqrt{3}\rho/2 \end{pmatrix}$$

The entries of this matrix $R_v$ are computed as follows:
\[ v(0) = \frac{\sqrt{3}}{2} u(0) + \frac{1}{2} u(1) \]

\[ E[v(0)^2] = \frac{3}{4} + \frac{1}{4} + 2 \frac{\sqrt{3}}{4} \rho \]

etc. Now if we compare the energy of the two coordinates:

\[ \sigma_{v(0)}^2 = 1 + \sqrt{3} \rho / 2 \]

\[ \sigma_{v(1)}^2 = 1 - \sqrt{3} \rho / 2 \]

we see that the total average energy is still 2, but now the energy in \( v(0) \) is greater than in \( v(1) \). If \( \rho = 0.95 \), then 91.1\% of the total average energy has been packed in the first sample.

\[ \sigma_{v(0)}^2 = 1.82 \quad \sigma_{v(1)}^2 = 0.18 \]

The correlation between \( v(0) \) and \( v(1) \) is given by

\[ \rho_{v(0,1)} = \frac{E[v(0)v(1)]}{\sigma_{v(0)} \sigma_{v(1)}} \]

\[ = \frac{\rho}{2\sqrt{1 - \frac{3}{4} \rho^2}} \]

which is less in absolute value than \( |\rho| \) for \( |\rho| < 1 \). For \( \rho = 0.95 \), we find \( \rho_{v(0,1)} = 0.83 \). Hence the correlation between the transform coefficients has been reduced.

Suppose instead we use the transform matrix

\[ A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

then we find

\[ \sigma_{v(0)}^2 = 1 + \rho \]

\[ \sigma_{v(1)}^2 = 1 - \rho \]

\[ \rho_{v(0,1)} = 0 \]

For \( \rho = 0.95 \), now 97.5\% of the energy is packed into \( v(0) \), and the two components \( v(0) \) and \( v(1) \) are uncorrelated.
DCT basis images for 8x8 blocks: